10. Linear Algebra

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1 Determinants (and singularity)

Putnam 2008/A2. Alan and Barbara take turns filling entries of an initially empty 2008×2008 array. Alan plays first. At each turn, a player places a real number in a vacant entry. When all entries are filled, Alan wins if the determinant of the resulting matrix is nonzero; Barbara wins if it is zero. Which player has a winning strategy?

VTRMC 2004/1. Let I denote the 2×2 identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; let A, B, and C be arbitrary 2×2 real matrices; finally, define the 4×4 matrices M and N by $M = \begin{bmatrix} I & A \\ B & C \end{bmatrix}$ and $N = \begin{bmatrix} I & B \\ A & C \end{bmatrix}$. Is it possible that exactly one of M and N is invertible?

VTRMC 2007/6. Let A and B be symmetric real $n \times n$ matrices. Suppose there are $n \times n$ matrices X, Y such that $\det(AX + BY) \neq 0$. Prove that $\det(A^2 + B^2) \neq 0$.

2 Thinking in Matrices

Well-known result. A matrix A is *symmetric* if $A^T = A$ and *skew-symmetric* if $A^T = -A$. Prove that any matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

Putnam 1991/A2. Let A and B be distinct $n \times n$ real matrices. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

VTRMC 2010/1, modified. Let A be a $n \times n$ integer matrix. Suppose

$$I + A + A^2 + \cdot + A^{100} = 0.$$

Show that $A^k + A^{k+1} + \cdots + A^{100}$ has determinant ± 1 for every positive integer $k \leq 100$.

3 Rank and dimension

Putnam 2003/B1. Do there exist polynomials $a(x), b(x), c(y), d(y)$ such that, for all x and y,

$$1 + xy + x^2y^2 = a(x)c(y) + b(x)d(y)?$$

VTRMC 2005/7. Let A be a 5×10 real matrix. Suppose every 5×1 real matrix (i.e. column vector in 5 dimensions) can be written in the form Au, where u is a 10×1 real matrix. Prove that every 5×1 real matrix can be written in the form AA^Tv where v is a 5×1 real matrix.

Putnam 2009/B4. Say that a polynomial with real coefficients in two variables, x, y, is *balanced* if the average value of the polynomial on each circle centered at the origin is 0. The balanced polynomials of degree at most 2009 form a vector space V over $\mathbb{R}$. Find the dimension of V.