9. Recursions

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1 Classical results

Tilings. Determine the number of ways to tile a 1×10 strip using only 1×1 or 1×2 tiles.

Catalan numbers. Find a closed-form expression for the number of valid sequences containing n pairs of parantheses. For example, when $n = 2$, there are 2 valid sequences: ()() and (()). The sequence ())(is not valid.

2 Problems

VTRMC 2011/0. Go to **Gates 4307** at 8:45am on Saturday.

VTRMC 2008/2. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are {1, 3, 3, 3, 3, 3} and {1, 3, 1, 3, 1, 3, 1, 3}.)

USAMO 1996/4. A type 1 sequence is a sequence with each term 0 or 1 which does not have 0, 1, 0 as consecutive terms. A type 2 sequence is a sequence with each term 0 or 1 which does not have 0, 0, 1, 1 or 1, 1, 0, 0 as consecutive terms. Show that there are twice as many type 2 sequences of length $n + 1$ as type 1 sequences of length n.

VTRMC 2001/3. For each positive integer n, let S_n denote the total number of squares in an $n \times n$ square grid. Thus $S_1 = 1$ and $S_2 = 5$, because a 2×2 square grid has four 1×1 squares and one 2×2 square. Find a recurrence relation for S_n, and use it to calculate the total number of squares on a chess board (i.e. determine S_8).

Famous. How about the number of rectangles?

GA 18. Prove that for any $n \geq 1$, a $2^n \times 2^n$ checkerboard with any 1×1 square removed can be tiled by L-shaped triominoes.

Putnam 2007/B3. Let $x_0 = 1$, and for $n \geq 0$, let

$$x_{n+1} = 3x_n + \lfloor x_n \sqrt{5} \rfloor.$$

In particular, $x_1 = 5$, $x_2 = 26$, $x_3 = 136$, $x_4 = 712$. Find a closed-form expression for x_{2007}.