8. Convergence

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1 Classical results

Harmonic series. Without using Calculus, show that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Alternating series. Let (a_n) be a monotonic decreasing sequence of positive real numbers. Then the series $\sum_{n=1}^{\infty}(-1)^n a_n$ is convergent.

2 Problems

VTRMC 2002/7. Let $(a_n)_{n>1}$ be an infinite sequence with $a_n > 0$ for all n. For $n > 1$, let b_n denote the geometric mean of $a_1, \ldots, a_n$, that is, $\sqrt[n]{a_1 \cdots a_n}$. Suppose $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that $\sum_{n=1}^{\infty} b_n^2$ is also convergent.

VTRMC 2006/5. Let (a_n) be a monotonic decreasing sequence of positive real numbers with limit 0 (so $a_1 \geq a_2 \geq \cdots \geq 0$). Let (b_n) be a rearrangement of the sequence such that for every non-negative integer m, the terms b_{3m+1}, b_{3m+2}, b_{3m+3} are a rearrangement of the terms a_{3m+1}, a_{3m+2}, a_{3m+3} (thus, for example, the first 6 terms of the sequence (b_n) could be $a_3, a_2, a_1, a_4, a_6, a_5$). Prove or give a counterexample to the following statement: the series $\sum_{n=1}^{\infty}(-1)^n b_n$ is convergent.

Putnam 2001/B6. Assume that $(a_n)_{n\geq 1}$ is an increasing sequence of positive real numbers such that $\lim \frac{a_n}{n} = 0$. Must there exist infinitely many positive integers n such that $a_{n-i} + a_{n+i} < 2a_n$ for $i = 1, 2, \ldots, n-1$?

VTRMC 2004/7. Let (a_n) be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$. Prove that $\sum_{n=1}^{\infty} \left|1 - \frac{a_{n+1}}{a_n}\right|$ is divergent.