4. Polynomials

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CMU Putnam Seminar, Fall 2011

1 Classical results

Well-known fact. Let $P(n)$ be a polynomial with integer coefficients, and let a and b be integers. Show that $P(a) - P(b)$ is divisible by $a - b$.

Lagrange Interpolation. Show that there is a degree-4 polynomial which takes values $P(0) = 0$, $P(1) = 0$, $P(2) = 0$, $P(3) = 1$, and $P(4) = 1$.

Eisenstein. Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial, such that there is a prime p for which

(i) p divides each of $a_0, a_1, \ldots, a_{n-1}$,

(ii) p does not divide a_n, and

(iii) p^2 does not divide a_0.

Then $P(x)$ cannot be expressed as the product of two non-constant polynomials with integer coefficients.

2 Problems

GA 173. Let $a_1, \ldots, a_n$ be positive real numbers. Prove that the polynomial $P(x) = x^n - a_1 x^{n-1} - a_2 x^{n-2} - \cdots - a_n$ has a unique positive zero.

IMO 1993/1. Let $P(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $P(x)$ cannot be expressed as the product of two non-constant polynomials with integer coefficients.

GA 183 (Classical). Prove that for every prime number p, the polynomial

$$P(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

cannot be expressed as the product of two non-constant polynomials with integer coefficients.

USAMO 1995/4, modified. Suppose $q_0, q_1, q_2, \ldots$ is an infinite sequence of integers statisfying the following two conditions:

(i) $m - n$ divides $q_m - q_n$ for $m > n \geq 0$,

(ii) there is a polynomial P and an integer Δ such that $|q_n - P(n)| < \Delta$ for all n.

Prove that there is a polynomial Q such that $q_n = Q(n)$ for all n.

Lucas. The zeros of the derivative $P'(z)$ of any polynomial lie in the convex hull of the zeros of the polynomial $P(z)$.