1. Proof by contradiction

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1 Classical results

1. A real-valued function f is called *convex* if for every real $0 < \lambda < 1$ and every real x, y, we have

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)\,.$$

   Suppose that f is convex, and fix an arbitrary real x_0. Show that for all $y > x_0$, the function

$$g(y) = \frac{f(y) - f(x_0)}{y - x_0}$$

   is increasing in y. In other words, show that the slope of the secant line through $(x_0, f(x_0))$ increases as we move to the right.

2. Recall that

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots \approx 2.718\ldots$$

   Prove that e is irrational.

2 Problems

Hungary 1999 (Gelca-Andreescu 10). Let $n > 1$ be an arbitrary positive integer, and let k be the number of positive prime numbers less than or equal to n. Select $k+1$ positive integers such that none of them divides the product of all the others. Prove that there exists a number among the chosen $k+1$ that is bigger than n.

Germany 1985 (Gelca-Andreescu 4). Every point in $\mathbb{R}^3$ is colored either red, green, or blue. Prove that one of the colors attains all distances, i.e., every positive real number represents the distance between two points of this color.

IMO 2001/4. Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i\,.$$

Prove that there are two different permutations b and c such that $n!$ is a divisor of $S(b) - S(c)$.

USAMO 2000/1. Call a real-valued function *very convex* if:

$$\frac{f(x)+f(y)}{2} \ge f\left(\frac{x+y}{2}\right) + |x-y|$$

holds for all real numbers x and y. Prove that no *very convex* function exists.