Inequalities

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1 Problems

Putnam 2004/B2. Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m}\frac{n!}{n^n}.$$

Gelca and Andreescu 104. If $a_1 + \cdots + a_n = n$, prove that $a_1^4 + \cdots + a_n^4 \geq n$.

Gelca and Andreescu 105. Let $a_1, \ldots, a_n$ be distinct real numbers. Find the maximum of

$$a_1 a_{\sigma(1)} + a_2 a_{\sigma(2)} + \cdots a_n a_{\sigma(n)},$$

over all permutations of the set $\{1, \ldots, n\}$.

Gelca and Andreescu 109. Let $P(x)$ be a polynomial with positive real coefficients. Prove that

$$\sqrt{P(a)P(b)} \geq P(\sqrt{ab}),$$

for all positive real numbers a, b.

Putnam 2003/A2. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be nonnegative real numbers. Show that

$$(a_1 a_2 \cdots a_n)^{1/n} + (b_1 b_2 \cdots b_n)^{1/n} \leq [(a_1 + b_1)(a_2 + b_2) \cdots (a_n + b_n)]^{1/n}.$$

Putnam 2000/A1. Let A be a positive real number. What are the possible values of $\sum_{j=0}^{\infty} x_j^2$, given that $x_0, x_1, \ldots$ are positive numbers for which $\sum_{j=0}^{\infty} x_j = A$?

Putnam 2007/A2. Find the least possible area of a convex set in the plane which touches both branches of the hyperbola $xy = 1$ and both branches of $xy = -1$.

Putnam 2002/B3. Show that for all integers $n > 1$,

$$\frac{1}{2ne} < \frac{1}{e} - \left(1 - \frac{1}{n}\right)^n < \frac{1}{ne}.$$

2 Bonus problem

Gelca and Andreescu 83. Prove that every polynomial $P(x)$ that takes only nonnegative values can be written as the sum of the squares of two polynomials.