Polynomials

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21 September 2010

1 Problems

Gelca and Andreescu 155. Let a, b, c be real numbers. Show that $a, b, c \geq 0$ if and only if $a+b+c \geq 0$, $ab+bc+ca \geq 0$, and $abc \geq 0$.

Euler. Prove that there is no polynomial $P(x)$ with integer coefficients and degree at least 1, such that $P(0)$, $P(1)$, $P(2)$, $\ldots$ are all prime.

Putnam 2005/B1. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a.

Putnam 2007/B1. Let f be a nonconstant polynomial with positive integer coefficients. Prove that if k is a positive integer, then $f(k)$ divides $f(f(k)+1)$ if and only if $k = 1$.

Russia 2002 (GA 147). Let $P(x)$ be a polynomial of odd degree with real coefficients. Show that the polynomial $P(P(x)) = 0$ has at least as many real roots as the equation $P(x) = 0$, counted without multiplicities.

Romania 1979 (GA 153). Let $P(z)$ be a polynomial with complex coefficients. Prove that $P(z)$ is an even function if and only if there exists a polynomial $Q(z)$ with complex coefficients satisfying $P(z) = Q(z)Q(-z)$.

Putnam 2004/B1. Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers $c_n r$, $c_n r^2 + c_{n-1} r$, $c_n r^3 + c_{n-1} r^2 + c_{n-2} r$, $\ldots$, $c_n r^n + c_{n-1} r^{n-1} + \cdots c_1 r$ are integers.

2 Bonus problems

Putnam 2001/A3. For each integer m, consider the polynomial

$$P_m(x) = x^4 - (2m+4)x^2 + (m-2)^2.$$

For what values of m is $P_m(x)$ the product of two nonconstant polynomials with integer coefficients?

Putnam 2003/B4. Let

$$\begin{aligned} f(z) &= az^4 + bz^3 + cz^2 + dz + e \\ &= a(z-r_1)(z-r_2)(z-r_3)(z-r_4), \end{aligned}$$

where a, b, c, d, e are integers, $a \neq 0$. Show that if $r_1 + r_2$ is a rational number and $r_1 + r_2 \neq r_3 + r_4$, then $r_1 r_2$ is a rational number.

Putnam 2007/B5. Let k be a positive integer. Prove that there exist polynomials $P_0(n)$, $P_1(n)$, $\ldots$, $P_{k-1}(n)$ (which may depend on k) such that for any integer n,

$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$