Induction

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1 Problems

From Gelca and Andreescu. Finitely many lines divide the plane into many regions. Show that these regions can be colored black or white such that neighboring regions have different colors. (Here, two regions are considered to be neighboring if they share a full edge, not just a corner.)

Putnam 2002/B1. Shanille O'Keal shoots free throws on a basketball court. She hits the first and misses the second, and thereafter the probability that she hits the next shot is equal to the proportion she has hit so far. What is the probability she hits exactly 50 of her first 100 shots?

Generalized Polya Urn. An urn starts with one red ball and one green ball. The urn will always contain a square number of red balls and a square number of blue balls, and it is updated in the following way. Each second, one ball is drawn from the urn uniformly at random. Let x^2 be the number of balls of its color that were in the urn. Return it to the urn, and also add $2x + 1$ more balls of its same color. Formally, if there were r^2 red balls and g^2 green balls, then in the next second, the urn will contain:

- $(r + 1)^2$ red balls and g^2 green balls, with probability $\frac{r^2}{r^2+g^2}$, or
- r^2 red balls and $(g + 1)^2$ green balls, with probability $\frac{g^2}{r^2+g^2}$.

This process continues forever.

Show that with probability 1, there will be a finite time after which every future ball drawn is the same color.

Putnam 2003/B2. Let n be a positive integer. Starting with the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$, form a new sequence of $n - 1$ entries $\frac{3}{4}, \frac{5}{12}, \ldots, \frac{2n-1}{2n(n-1)}$, by taking the averages of two consecutive entries in the first sequence. Repeat the averaging of neighbors on the second sequence to obtain a third sequence of $n - 2$ entries, and continue until the final sequence produced consists of a single number x_n. Show that $x_n < \frac{2}{n}$.

Fermat's Little Theorem. Show that for every prime p and every integer x, we have $x^p \equiv x \pmod{p}$.

Putnam 2005/A1. Show that every positive integer is a sum of one or more numbers of the form $2^r 3^s$, where r and s are nonnegative integers and no summand divides another. (For example, 23 = 9 + 8 + 6.)

Putnam 2006/B3. Let S be a finite set of points in the plane. A linear partition of S is an unordered pair $\{A, B\}$ of subsets of S such that $A \cup B = S$, $A \cap B = \emptyset$, and A and B lie on opposite sides of some straight line disjoint from S (A or B may be empty). Let L_S be the number of linear partitions of S. For each positive integer n, find the maximum of L_S over all sets S of n points.