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Section

1. +4 $\frac{d}{dy}(y^3-y) = 3y^2-1 = 0 \Rightarrow y = \pm\sqrt{3}/3$

y^3-y at $y \in \{-1, -\sqrt{3}/3, \sqrt{3}/3, 1\} \Rightarrow \{0, (\frac{1}{9}-\frac{1}{3})\sqrt{3}, (\frac{1}{3}-\frac{1}{9})\sqrt{3}\}$

So $\forall\ y \in [-1,1]$, $y^3-y \leq \frac{2}{9}\sqrt{3} < 2$.

$$\int_{-1}^{1}\int_{y^3-y}^{2} y \, dx \, dy = \int_{-1}^{1} yx \Big|_{y^3-y}^{2} dy = \int_{-1}^{1} 2y - y^4 + y^2 \, dy$$

$$= (y^2 - y^5/5 + y^3/3)\Big|_{-1}^{1} = (1 - 1/5 + 1/3) - (1 + 1/5 - 1/3) = 4/15$$

2. a.) +4 Let $0 \leq x \leq 2$, $\max(1-x, 3x-3) \leq y \leq 1+x$

If $1-x \geq 3x-3$ then $4x \leq 4 \Rightarrow x \leq 1$

$\min_{0\leq x\leq 1} 1-x = 0$, $\min_{1<x\leq 2} 3x-3 = 0$. $\max_{0\leq x\leq 2} 1+x = 3$

So $0 \leq y \leq 3$.

Known $y \leq 1+x \Rightarrow x \geq \cancel{1+y}$ $y-1$,

$1-x \leq y \wedge 3x-3 \leq y \Rightarrow x \geq 1-y \wedge x \leq 1 + y/3$.

So $\max(\cancel{1+y}\ y-1, 1-y) \leq x \leq 1 + y/3$.

b.) Let $0 \leq y \leq 3$, $\max(1-y, y-1) \leq x \leq 1 + y/3$.

If $1-y \geq y-1$ then $2y \leq 2 \Rightarrow y \leq 1$.

$\min_{0\leq y\leq 1} 1-y = 0$, $\min_{1<y\leq 3} y-1 = 0$, $\max_{0\leq y\leq 3} 1 + y/3 = 2$.

So $0 \leq x \leq 2$.

Known $x \leq 1 + y/3 \Rightarrow y \geq 3x-3$.

$1-y \leq x \wedge y-1 \leq x \Rightarrow y \geq 1-x \wedge y \leq x+1$

So $\max(3x-3, 1-x) \leq y \leq x+1$

3. +3 Known: $\forall (x,y) \in R$ $0 \leq y \leq 4$, $\sqrt{y} \leq x \leq 2$

$\Rightarrow y \leq x^2$, $y \leq 4$, $y \geq 0$, $x \leq 2$, $x \geq \sqrt{y}$ Note $\sqrt{y} \geq 0$, $x^2 \leq 4$.

$\Rightarrow 0 \leq y \leq x^2$, $0 \leq x \leq 2$.

$$\int_0^2\int_0^{x^2} e^{x^3} dy \, dx = \int_0^2 e^{x^3} y\Big|_0^{x^2} dx = \int_0^2 x^2 e^{x^3} dx = \frac{1}{3}e^{x^3}\Big|_0^2$$

$$= \frac{1}{3}e^8 - \frac{1}{3} = \frac{1}{3}(e^8 - 1)$$

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4. $x=0,\ y=0,\ z=x,\ x^2+y^2+z^2=4$ forms the boundary of R.
$x \geq 0$ and $y \geq 0$, or otherwise R would not be in the first octant.
$x^2+y^2+z^2 \leq 4$, or otherwise $\{(t,0,t) \mid t>\sqrt{2}\} \subset R$ would be an unbounded subset.
$z \geq x$, or otherwise $(0,0,-1) \in R$ would not be in the first octant.

a.) y bounds: $0,\ \max_{x,z}\sqrt{4-x^2-z^2},\ \min_{x,z}\sqrt{4-x^2-z^2}$, Since $z \geq x \geq 0$ &
$x^2+z^2=4-y^2 \leq 4$, we know $\min_{x,z}\sqrt{4-x^2-z^2} \geq 0$.
So $0 \leq y \leq \max_{x,z}\sqrt{4-x^2-z^2} \Rightarrow 0 \leq y \leq 2$ $\quad (x,y,z)=(0,2,0)$.
x bounds: $0,\ \max_z\sqrt{4-y^2-z^2},\ \min_z\sqrt{4-y^2-z^2}$, Since $y \geq 0$ and $x \geq 0$,
we know $y^2+z^2=4-x^2 \leq 4$, so $\min_z\sqrt{4-y^2-z^2} \geq 0$
To find $\max_z\sqrt{4-y^2-z^2}$, set $z=x$ so $x=\sqrt{4-y^2-x^2}$
$\Rightarrow 2x^2=4-y^2 \Rightarrow x=\sqrt{\frac{1}{2}(4-y^2)}$
So $0 \leq x \leq \sqrt{\frac{1}{2}(4-y^2)}$
z bounds: $x,\ \sqrt{4-x^2-y^2}$
$\sqrt{4-x^2-y^2} = \sqrt{\frac{1}{2}(4-y^2)} \geq x$ so $x \leq z \leq \sqrt{4-x^2-y^2}$.
$\therefore \iiint_R f\,dV = \int_0^2 \int_0^{\sqrt{\frac{1}{2}(4-y^2)}} \int_x^{\sqrt{4-x^2-y^2}} f(x,y,z)\,dy\,dz\,dx$.

b.) x bounds: $0,\ \max_{y,z}\sqrt{4-y^2-z^2},\ \min_{y,z}\sqrt{4-y^2-z^2}$. Since $y \geq 0,\ z \geq x \geq 0$
and $y^2+z^2=4-x^2 \leq 4$, we know $\min_{y,z}\sqrt{4-y^2-z^2} \geq 0$.
So $0 \leq x \leq \max_{y,z}\sqrt{4-y^2-z^2}$. To find $\max_{y,z}\sqrt{4-y^2-z^2}$, set $z=x$
and $y=0$, so $x=\sqrt{4-x^2} \Rightarrow 2x^2=4 \Rightarrow x=\sqrt{2}$ or $\cancel{-\sqrt{2}}$
So $0 \leq x \leq \sqrt{2}$.
z bounds: $x,\ \max_y\sqrt{4-x^2-y^2},\ \min_y\sqrt{4-x^2-y^2}$. Since $y \leq \sqrt{4-x^2-z^2}$,
$\min_y\sqrt{4-x^2-y^2} \geq \sqrt{z^2}=z \geq x$, $\max_y\sqrt{4-x^2-y^2}$ is attained by
$y=0 \Rightarrow x \leq z \leq \sqrt{4-x^2}$
y bounds: $0,\ \sqrt{4-x^2-z^2}$. Since $z \geq x \geq 0$, $x^2+z^2=4-y^2 \leq 4$
so $\sqrt{4-x^2-z^2} \geq 0$ so $0 \leq y \leq \sqrt{4-x^2-z^2}$
$\therefore \iiint_R f\,dV = \int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} \int_0^{\sqrt{4-x^2-z^2}} f(x,y,z)\,dy\,dz\,dx$

5. $y=0,\ y=x,\ y=2-x,\ z=y,\ z=2-y$ form the boundary of R

$x \leftarrow y \leftarrow z$ Do $dz\,dy\,dx$. would've been easier to do $dz\,dx\,dy$

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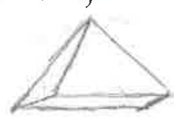
Turns out the only bounded region is $\{(x,y,z) \mid y \geq 0,\ y \leq x,\ y \leq 2-x,\ z \geq y,\ z \leq 2-y\}$.

It looks like:



x bounds: $x \geq y \geq 0$ and $x \leq 2-y \leq 2$. so $0 \leq x \leq 2$

y bounds: $y \geq 0$ and $y \leq x$ and $y \leq 2-x$

Both x and $2-x$ are non-negative.

If $x \leq 2-x \Rightarrow 2x \leq 2 \Rightarrow x \leq 1$

So $0 \leq y \leq x$ for $0 \leq x \leq 1$, and $0 \leq y \leq 2-x$ for $1 < x \leq 2$.

z bounds: $z \geq y$ and $z \leq 2-y$.

For $0 \leq y \leq x \leq 1$ and $0 \leq y \leq 2-x$, $1 < x \leq 2$

we have $0 \leq y \leq 1$, so $2-y \geq y$

So $y \leq z \leq 2-y$

$$\iiint_R y\,dV = \int_0^1\int_0^x\int_y^{2-y} y\,dz\,dy\,dx + \int_1^2\int_0^{2-x}\int_y^{2-y} y\,dz\,dy\,dx$$
$$= \int_0^1\int_0^x yz\Big|_y^{2-y} dy\,dx + \int_1^2\int_0^{2-x} yz\Big|_y^{2-y} dy\,dx$$
$$= \int_0^1\int_0^x 2y-2y^2\,dy\,dx + \int_1^2\int_0^{2-x} 2y-2y^2\,dy\,dx$$
$$= \int_0^1 y^2-\tfrac{2}{3}y^3\Big|_0^x dx + \int_1^2 y^2-\tfrac{2}{3}y^3\Big|_0^{2-x} dx$$
$$= \int_0^1 x^2-\tfrac{2}{3}x^3\,dx + \int_1^2 (2-x)^2-\tfrac{2}{3}(2-x)^3\,dx$$
$$= x^3/3 - x^4/6\Big|_0^1 + -(2-x)^3/3 + (2-x)^4/6\Big|_1^2$$
$$= 1/3 - 1/6 + 1/3 - 1/6 = 1/3$$

6. 

+4

For a given $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ $r\sin\theta \leq 2 - r\cos\theta$

$\Rightarrow r \leq 2/(\sin\theta + \cos\theta)$. The expression is non-negative and well-defined since $\sin\theta + \cos\theta = \cos(\theta - \frac{\pi}{4})$ which is positive in $\theta \in [\frac{\pi}{4}, \frac{\pi}{2}]$. Also, $r > 0$

$$\int_{\pi/4}^{\pi/2}\int_0^{2/(\sin\theta+\cos\theta)} \frac{r\cos\theta - r\sin\theta}{r}\, r\,dr\,d\theta = \int_{\pi/4}^{\pi/2}\int_0^{2/(\sin\theta+\cos\theta)} r(\cos\theta - \sin\theta)\,dr\,d\theta$$
$$= \int_{\pi/4}^{\pi/2} (\cos\theta - \sin\theta)\frac{r^2}{2}\Big|_0^{2/(\sin\theta+\cos\theta)} d\theta$$

$= \int_{\pi/4}^{\pi/2} 2(\cos\theta - \sin\theta) / (\sin\theta + \cos\theta)^2 \, d\theta$

Let $u = (\sin\theta + \cos\theta)$. $du = (\cos\theta - \sin\theta)\, d\theta$

$u_{low} = \sqrt{2}$ $u_{high} = 1$

$= \int_{\sqrt{2}}^{1} 2/u^2 \, du = -2/u \Big|_{\sqrt{2}}^{1} = -2 + \sqrt{2}$