21-241 – Solution to Homework assignment week #8

Laurent Dietrich
Carnegie Mellon University, Fall 2016, Sec. F and G

Ex 1

1. Let us prove that AB invertible $\Rightarrow$ B invertible by contraposition. So let us assume B is not invertible. Then there exists $x \neq 0$ such that $Bx = 0$. But then also $ABx = A0 = 0$, which means that AB is not invertible.

   For A we work in the direct sense and use the definition of invertible. Since now we know that B is invertible and we assumed that AB is invertible, there exists C invertible such that
   $$ABC = I_n$$
   Multiplying to the right by $C^{-1}B^{-1}$ we get
   $$A = C^{-1}B^{-1}$$
   which is invertible.

2. By contraposition of the above implication : if A or B is not invertible, then AB is not invertible.

   Remark : actually we proved that in the definition of invertible, asking for $AA' = A'A = I$ is too much, we can ask for only one of these relations and the other will follow. This would simplify this exercise a lot (write that $ABC = CAB = I$ for some C, then A and B are invertible with resp. inverses BC and CA)... some of you did that, it is fine as well, but do have in mind that this property you used is not easy !

Ex 2

1. $\det(B^m) = \det(0) = 0$ but also $\det(B^m) = \det(B)^m$ so necessarily $\det(B) = 0$.
2. $\det(A^2) = \det(A)$ since $A = A^2$ but also $\det(A^2) = \det(A)^2$. So
   $$\det(A) = \det(A)^2$$
   and in the end $\det(A) = 0$ or 1.

Ex 3

The determinant of A can be factorized as
$$-k^2(k^2 + k - 2) = -k^2(k+2)(k-1)$$
so A is invertible for all $k \neq 0, -2, 1$.

Ex 4

We compute and expand

$$\det(A - \lambda I_3) = -\lambda^3 + 2\lambda^2 + \lambda - 2$$

$\lambda = 1$ is clearly a solution so we can factorize by $(\lambda - 1)$ and get

$$\det(A - \lambda I_3) = -(\lambda - 1)(\lambda^2 - \lambda - 2) = -(\lambda - 1)(\lambda + 1)(\lambda - 2)$$

So the eigenvalues of A are $1, -1$ and 2. We now compute bases of the associated eigenspaces by usual row reduction :

$$E_1 = \text{null}(A - I) = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right) \qquad E_{-1} = \text{null}(A + I) = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}\right)$$

$$E_2 = \text{null}(A - 2I) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

Observe that A has three independent eigenvectors so we are in the nice situation we described during Wednesday's lecture : these eigenvectors form a basis of $\mathbb{R}^3$, so every $x \in \mathbb{R}^3$ can be uniquely decomposed along these eigenvectors and doing this will simplify a lot all the computations involving A. We say that A is *diagonalizable*.