NECESSARY AND SUFFICIENT CONDITIONS IN THE PROBLEM
OF OPTIMAL INVESTMENT IN INCOMPLETE MARKETS

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Following Ann. Appl. Probab. 9 (1999) 904–950 we continue the study of the problem of expected utility maximization in incomplete markets. Our goal is to find minimal conditions on a model and a utility function for the validity of several key assertions of the theory to hold true. In the previous paper we proved that a minimal condition on the utility function alone, that is, a minimal market independent condition, is that the asymptotic elasticity of the utility function is strictly less than 1. In this paper we show that a necessary and sufficient condition on both, the utility function and the model, is that the value function of the dual problem is finite.

1. Introduction and main results. We study the same financial framework as in [10] and refer to this paper for more details and references. We consider a model of a security market which consists of \(d + 1\) assets, one bond and \(d\) stocks. We work in discounted terms, that is, we suppose that the price of the bond is constant, and denote by \(S = (S^i)_{1 \leq i \leq d}\) the price process of the \(d\) stocks. The process \(S\) is assumed to be a semimartingale on a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\). Here \(T\) is a finite time horizon. To simplify notation we assume that \(\mathcal{F} = \mathcal{F}_T\).

A (self-financing) portfolio \(\Pi\) is defined as a pair \((x, H)\), where the constant \(x\) is the initial value of the portfolio, and \(H = (H^i)_{1 \leq i \leq d}\) is a predictable \(S\)-integrable process, where \(H^i_t\) specifies how many units of asset \(i\) are held in the portfolio at time \(t\). The value process \(X = (X_t)_{0 \leq t \leq T}\) of such a portfolio \(\Pi\) is given by

\[
X_t = X_0 + \int_0^t H_u dS_u, \quad 0 \leq t \leq T.
\]

We denote by \(\mathcal{X}(x)\) the family of wealth processes with nonnegative capital at any instant, that is, \(X_t \geq 0\) for all \(t \in [0, T]\), and with initial value equal to \(x\). In other words

\[
\mathcal{X}(x) = \{X \geq 0 : X \text{ is defined by (1) with } X_0 = x\}.
\]

We shall use the shorter notation \(\mathcal{X}\) for \(\mathcal{X}(1)\). Clearly,

\[
\mathcal{X}(x) = x\mathcal{X} = \{xX : X \in \mathcal{X}\} \quad \text{for } x \geq 0.
\]
A probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an *equivalent local martingale measure* if any $X \in \mathcal{X}$ is a local martingale under $\mathbb{Q}$. The family of equivalent local martingale measures will be denoted by $\mathcal{M}$. We assume throughout that

$$\mathcal{M} \neq \emptyset.$$  

This condition is intimately related to the absence of arbitrage opportunities on the security market. See [4, 5] for precise statements and references.

We also consider an economic agent in our model, whose preferences are modeled by a utility function $U : (0, \infty) \to \mathbb{R}$ for wealth at maturity time $T$. Hereafter we will assume that the function $U$ is strictly increasing, strictly concave, continuously differentiable and satisfies the Inada conditions

$$U'(0) = \lim_{x \to 0} U'(x) = \infty, \quad U'(-\infty) = \lim_{x \to -\infty} U'(x) = 0.$$  

For a given initial capital $x > 0$, the goal of the agent is to maximize the expected value of terminal utility. The value function of this problem is denoted by

$$u(x) = \sup_{X \in \mathcal{X}(x)} \mathbb{E}[U(X_T)].$$  

Intuitively speaking, the value function $u$ plays the role of the utility function of the investor at time 0, if she subsequently invests in an optimal way. A well-known tool in studying the optimization problem (4) is the use of duality relationships in the spaces of convex functions and semimartingales; see, for example, [1–3, 6–11, 13].

The conjugate function $V$ to the utility function $U$ is defined as

$$V(y) = \sup_{x > 0} [U(x) - xy], \quad y > 0.$$  

It is well known (see, e.g., [12]) that if $U$ satisfies the hypotheses stated above, then $V$ is a continuously differentiable, decreasing, strictly convex function satisfying $V'(0) = -\infty$ and $V'(-\infty) = 0$, $V(0) = U(\infty)$, $V(\infty) = U(0)$, and the following relation holds true:

$$U(x) = \inf_{y > 0} [V(y) + xy], \quad x > 0.$$  

In addition the derivative of $U$ is the inverse function of the negative of the derivative of $V$, that is,

$$U'(x) = y \iff x = -V'(y).$$  

Further, we define the family $\mathcal{Y}$ of nonnegative semimartingales, which is dual to $\mathcal{X}$ in the following sense:

$$\mathcal{Y} = \{ Y \geq 0 : Y_0 = 1 \text{ and } XY \text{ is a supermartingale for all } X \in \mathcal{X} \}.$$
Note that, as $1 \in \mathcal{X}$, any $Y \in \mathcal{Y}$ is a supermartingale. Note also that the set $\mathcal{Y}$ contains the density processes of all $\mathbb{Q} \in \mathcal{M}$. For $y > 0$, we define 
\[ \mathcal{Y}(y) = y\mathcal{Y} = \{ yY : Y \in \mathcal{Y} \} \]
and consider the following optimization problem:
\[ v(y) = \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y_T)]. \]

The next result from [10] shows that the value functions $u$ and $v$ to the optimization problems (4) and (6) are conjugate.

**Theorem 1** ([10], Theorem 2.1). Assume that (2) and (3) hold true and (7)
\[ u(x) < \infty \quad \text{for some } x > 0. \]
Then:
1. $u(x) < \infty$ for all $x > 0$, and there exists $y_0 \geq 0$ such that $v(y)$ is finitely valued for $y > y_0$. The value functions $u$ and $v$ are conjugate
\[ v(y) = \sup_{x > 0} [u(x) - xy], \quad y > 0, \]
\[ u(x) = \inf_{y > 0} [v(y) + xy], \quad x > 0. \]

The function $u$ is continuously differentiable on $(0, \infty)$ and the function $v$ is strictly convex on $\{v < \infty\}$.

The functions $u'$ and $v'$ satisfy
\[ u'(0) = \lim_{x \to 0} u'(x) = \infty, \]
\[ v'(\infty) = \lim_{y \to \infty} v'(y) = 0. \]

2. The optimal solution $\hat{Y}(y) \in \mathcal{Y}(y)$ to (6) exists and is unique provided that $v(y) < \infty$.

As in [10] we are interested in the following questions related to the optimization problems (4) and (6):
1. Does the optimal solution $\hat{X} \in \mathcal{X}(x)$ to (4) exist?
2. Does the value function $u(x)$ satisfy the usual properties of a utility function, that is, is it increasing, strictly concave, continuously differentiable and such that $u'(0) = \infty, u'(\infty) = 0$?
3. Does the dual value function $v$ have the representation
\[ v(y) = \inf_{\mathbb{Q} \in \mathcal{M}} \mathbb{E} \left[ V \left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \mathbb{I} \left( Y \in \mathcal{Y}(y) \right) \right]. \]

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ denotes the Radon–Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{P}$ on $(\Omega, \mathcal{F}) = (\Omega, \mathcal{F}_T)$?
In [10] (see Theorem 2.2 and the counterexamples in Section 5) we proved that a minimal assumption on the utility function $U$, which implies positive answers to these questions for an arbitrary financial model, is the condition on the asymptotic behavior of the elasticity of $U$,

$$AE(U) \triangleq \limsup_{x \to \infty} \frac{xU''(x)}{U(x)} < 1.$$ 

The subsequent theorem, which is the main result of the present paper, and Note 1 below imply that a necessary and sufficient condition for all three assertions to have positive answers in the framework of a particular financial model is the finiteness of the dual value function.

**THEOREM 2.** Assume that (2) and (3) hold true and

(10) \[ v(y) < \infty \quad \forall \ y > 0. \]

Then in addition to the assertions of Theorem 1 we have the following:

1. The value functions $u$ and $-v$ are continuously differentiable, increasing and strictly concave on $(0, \infty)$ and satisfy

$$u'(\infty) = \lim_{x \to \infty} u'(x) = 0,$$

$$-v'(0) = \lim_{y \to 0} -v'(y) = \infty.$$ 

2. The optimal solution $\hat{X}(x) \in \mathcal{X}(x)$ to (4) exists, for any $x > 0$, and is unique. In addition, if $y = u'(x)$ then

$$U'(\hat{X}_T(x)) = \hat{Y}_T(y),$$

where $\hat{Y}(y) \in \mathcal{Y}(y)$ is the optimal solution to (6). Moreover, the process $\hat{X}(x)\hat{Y}(y)$ is a martingale.

3. The dual value function $v$ satisfies (9).

**PROOF.** Theorem 2 is a rather straightforward consequence of its “abstract version,” Theorem 4. Admitting Theorem 4 as well as Proposition 1, the proof of Theorem 2 proceeds as follows.

For $x > 0$ and $y > 0$, let

(11) \[ \mathcal{C}(x) = \{ g \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \leq g \leq X_T, \ \text{for some} \ X \in \mathcal{X}(x) \}, \]

(12) \[ \mathcal{D}(y) = \{ h \in L^0(\Omega, \mathcal{F}, \mathbb{P}) : 0 \leq h \leq Y_T, \ \text{for some} \ Y \in \mathcal{Y}(y) \}. \]

In other words, $\mathcal{C}(x)$ and $\mathcal{D}(y)$ are the sets of random variables dominated by the final values of elements from $\mathcal{X}(x)$ and $\mathcal{Y}(y)$, respectively. With this notation, the value functions $u$ and $v$ take the form

$$u(x) = \sup_{g \in \mathcal{C}(x)} \mathbb{E}[U(g)],$$

$$v(y) = \inf_{h \in \mathcal{D}(y)} \mathbb{E}[V(h)].$$
According to Proposition 3.1 in [10] the sets \( C(x), x > 0 \), and \( D(y), y > 0 \), satisfy the conditions (16)–(18). Hence Theorem 4 implies assertions 1 and 2 of Theorem 2, except for the claim that the product \( \hat{X}(x)\hat{Y}(y) \) is a martingale. To prove the martingale property, note that \( \hat{X}(x)\hat{Y}(y) \) is a positive supermartingale [by the construction of the set \( \mathcal{Y}(y) \)] and that we obtain the following equality from item 2 of Theorem 4:

\[
E[\hat{X}_T(x)\hat{Y}_T(y)] = xy = \hat{X}_0(x)\hat{Y}_0(y).
\]

This readily implies the martingale property of \( \hat{X}(x)\hat{Y}(y) \).

To prove the final assertion 3, we use Proposition 1. We denote by \( \tilde{D} \) the set of Radon–Nikodym derivatives of equivalent martingale measures

\[
\tilde{D} = \left\{ h = \frac{dQ}{dP}, \ Q \in \mathcal{M} \right\}.
\]

The set \( \tilde{D} \) is closed under countable convex combinations. In addition,

\[
g \in C \iff g \geq 0 \quad \text{and} \quad E_Q[g] \leq 1 \quad \forall Q \in \mathcal{M}
\]

by the general duality relationships between the terminal values of strategies and the densities of equivalent martingale measures (see [4] and [5]). Hence the set \( \tilde{D} \) satisfies the assumptions of Proposition 1 and the result follows.

**NOTE 1.** In view of the duality relation (8), condition (10) is equivalent to

\[
u'(\infty) = \lim_{x \to \infty} u'(x) = 0,
\]

which may equivalently be restated as

\[
\lim_{x \to \infty} \frac{u(x)}{x} = 0.
\]

In particular, this shows the necessity of (10) for Theorem 2 to hold true.

**NOTE 2.** In [10], Theorem 2.2, we proved that the assertions of Theorem 2 follow from the assumptions of Theorem 1 and the condition \( AE(U) < 1 \) on the asymptotic elasticity of \( U \). Let us now deduce this result as an easy consequence of Theorem 2.

We need to show that \( AE(U) < 1 \) implies that \( v(y) < \infty \) for all \( y > 0 \). By Theorem 1 there is \( y_0 > 0 \) such that

\[
v(y) < \infty, \quad y > y_0.
\]

Further, the condition \( AE(U) < 1 \) is equivalent to the following property of \( V \) (see Lemma 6.3 in [10]): there are positive constants \( c_1 \) and \( c_2 \) such that

\[
V\left(\frac{y}{2}\right) \leq c_1 V(y) + c_2, \quad y > 0.
\]

The finiteness of \( v \) now follows from (13) and (14).
NOTE 3. Condition (10) may also be stated in the following equivalent form:

\[
\inf_{Q \in M} E \left[ V \left( y \frac{dQ}{dP} \right) \right] < \infty \quad \forall \ y > 0.
\]

Indeed, the implication (15) \( \Rightarrow \) (10) is trivial, as the density processes of martingale measures belong to \( \mathcal{Y} \). The more difficult reverse implication follows from Theorem 2.

2. The abstract version of the theorem. Let \( C \) and \( D \) be nonempty sets of positive random variables such that

1. The set \( C \) is bounded in \( L^0(\Omega, \mathcal{F}, P) \) and contains the constant function \( g = 1 \),

\[
\lim_{n \to \infty} \sup_{g \in C} P[|g| \geq n] = 0,
\]

\[
1 \in C.
\]

2. The sets \( C \) and \( D \) satisfy the bipolar relations

\[
g \in C \iff g \geq 0 \quad \text{and} \quad E[gh] \leq 1 \quad \forall h \in D,
\]

\[
h \in D \iff h \geq 0 \quad \text{and} \quad E[gh] \leq 1 \quad \forall g \in C.
\]

For \( x > 0 \) and \( y > 0 \), we define the sets

\[
C(x) = xC = \{xg : g \in C\},
\]

\[
D(y) = yD = \{yh : h \in D\},
\]

and the optimization problems

\[
u(x) = \sup_{g \in C(x)} E[U(g)],
\]

\[
v(y) = \inf_{h \in D(y)} E[V(h)].
\]

Here \( U = U(x) \) and \( V = V(y) \) are the functions defined in Section 1. If \( C(x) \) and \( D(y) \) are defined by (11) and (12), these value functions coincide with the value functions defined in (4) and (6).

Let us recall the following result from [10], which is the abstract version of Theorem 1.

THEOREM 3 (Theorem 3.1 in [10]). Assume that the sets \( C \) and \( D \) satisfy (16)–(18). Assume also that the utility function \( U \) satisfies (3) and that

\[
u(x) < \infty \quad \text{for some} \ x > 0.
\]
Then:

1. \( u(x) < \infty \) for all \( x > 0 \), and there exists \( y_0 \geq 0 \) such that \( v(y) \) is finitely valued for \( y > y_0 \). The value functions \( u \) and \( v \) are conjugate:

\[
\begin{align*}
    v(y) &= \sup_{x > 0} [u(x) - xy], \quad y > 0, \\
    u(x) &= \inf_{y > 0} [v(y) + xy], \quad x > 0.
\end{align*}
\]

(22)

The function \( u \) is continuously differentiable on \((0, \infty)\), and the function \( v \) is strictly convex on \( \{v < \infty\} \).

The functions \( u' \) and \( -v' \) satisfy

\[
\begin{align*}
    u'(0) &= \lim_{x \to 0} u'(x) = \infty, \\
    v'(\infty) &= \lim_{y \to \infty} v'(y) = 0.
\end{align*}
\]

2. If \( v(y) < \infty \), then the optimal solution \( \hat{h}(y) \in D(y) \) to (19) exists and is unique.

We now state the abstract version of Theorem 2. This theorem refines Theorem 3.2 in [10] in the sense that the condition \( AE(U) < 1 \) is replaced by the weaker condition (23) requiring the finiteness of the function \( v(y) \), for all \( y > 0 \).

**THEOREM 4.** Assume that the utility function \( U \) satisfies (3), the sets \( C \) and \( D \) satisfy (16)–(18), and that the value function \( v \) defined in (20) is finite

(23)

\[ v(y) < \infty \quad \forall \ y > 0. \]

Then, in addition to the assertions of Theorem 3, we have the following:

1. The value functions \( u \) and \( -v \) are continuously differentiable, increasing and strictly concave on \((0, \infty)\) and satisfy

\[
\begin{align*}
    u'(\infty) &= \lim_{x \to \infty} u'(x) = 0, \\
    -v'(0) &= \lim_{y \to 0} -v'(y) = \infty.
\end{align*}
\]

2. The optimal solution \( \hat{g}(x) \in C(x) \) to (19) exists, for all \( x > 0 \), and is unique. In addition, if \( y = u'(x) \), then

\[
U'(\hat{g}(x)) = \hat{h}(y),
\]

and

\[
E[\hat{g}(x)\hat{h}(y)] = xy,
\]

where \( \hat{h}(y) \in D(y) \) is the optimal solution to (20).

The proof of Theorem 4 is based on the following lemma.
**Lemma 1.** Assume that the set $\mathcal{C}$ satisfies (16)–(18) and the value function $u(x)$ defined in (19) is finite (for some or, equivalently, for all $x > 0$) and satisfies

$$\lim_{x \to \infty} \frac{u(x)}{x} = 0. \quad (24)$$

Then the optimal solution $\hat{g}(x) \in \mathcal{C}(x)$ exists for all $x > 0$.

**Proof.** The assertion that $u(x) < \infty$, for some $x > 0$, iff $u(x) < \infty$, for all $x > 0$, is a straightforward consequence of the concavity and monotonicity of $u$ and the fact that $u \geq U$. Also observe that, as remarked in Note 1, assertion (24) is equivalent to (23).

Fix $x > 0$. Let $(f^n)_{n \geq 1}$ be a sequence in $\mathcal{C}(x)$ such that

$$\lim_{n \to \infty} \mathbb{E}[U(f^n)] = u(x).$$

We can find a sequence of convex combinations $g^n \in \text{conv}(f^n, f^{n+1}, \ldots)$ which converges almost surely to a random variable $\hat{g}$ with values in $[0, \infty]$; see, for example, [4], Lemma A1.1. Since the set $\mathcal{C}(x)$ is bounded in $L^0(\Omega, \mathcal{F}, \mathbb{P})$, we deduce that $\hat{g}$ is almost surely finitely valued. By (18) and Fatou’s lemma, $\hat{g}$ belongs to $\mathcal{C}(x)$. We claim that $\hat{g}$ is the optimal solution to (19), that is,

$$\mathbb{E}[U(\hat{g})] = u(x).$$

Let us denote by $U^+$ and $U^-$ the positive and negative parts of the function $U$. From the concavity of $U$ we deduce that

$$\lim_{n \to \infty} \mathbb{E}[U(g^n)] = u(x)$$

and from Fatou’s lemma that

$$\liminf_{n \to \infty} \mathbb{E}[U^-(g^n)] \geq \mathbb{E}[U^-(\hat{g})].$$

The optimality of $\hat{g}$ will follow if we show that

$$\lim_{n \to \infty} \mathbb{E}[U^+(g^n)] = \mathbb{E}[U^+(\hat{g})]. \quad (25)$$

If $U(\infty) \leq 0$, then there is nothing to prove. So we assume that $U(\infty) > 0$.

The validity of (25) is equivalent to the uniform integrability of the sequence $(U^+(g^n))_{n \geq 1}$. If this sequence is not uniformly integrable then, passing if necessary to a subsequence still denoted by $(g^n)_{n \geq 1}$, we can find a constant $\alpha > 0$ and a disjoint sequence $(A^n)_{n \geq 1}$ of $(\Omega, \mathcal{F})$, that is,

$$A^n \in \mathcal{F}, \quad A^i \cap A^j = \emptyset \quad \text{if} \; i \neq j,$$

such that

$$\mathbb{E}[U^+(g^n) I(A^n)] \geq \alpha \quad \text{for} \; n \geq 1.$$
We define the sequence of random variables \((h^n)_{n \geq 1}\)
\[
h^n = x_0 + \sum_{k=1}^{n} g^k I(A^k),
\]
where
\[
x_0 = \inf\{x > 0 : U(x) \geq 0\}.
\]
For any \(f \in D\),
\[
E[h^nf] \leq x_0 + \sum_{k=1}^{n} E[g^k f] \leq x_0 + nx.
\]
Hence \(h^n \in C(x_0 + nx)\). On the other hand,
\[
E[U(h^n)] \geq \sum_{k=1}^{n} E[U^+(g^k)I(A^k)] \geq \alpha n,
\]
and therefore
\[
\limsup_{x \to \infty} \frac{u(x)}{x} \geq \limsup_{n \to \infty} \frac{E[U(h^n)]}{x_0 + nx} \geq \limsup_{n \to \infty} \frac{\alpha n}{x_0 + nx} = \alpha > 0.
\]
This contradicts (24). Therefore (25) holds true. □

PROOF OF THEOREM 4. Since, for \(x > 0\) and \(y > 0\),
\[
U(x) \leq V(y) + xy,
\]
and, for \(g \in C(x)\) and \(h \in D(y)\),
\[
E[gh] \leq xy,
\]
we have
\[
u(x) \leq v(y) + xy.
\]
In particular, the finiteness of \(v(y)\), for some \(y > 0\), implies the finiteness of \(u(x)\), for all \(x > 0\). It follows that the conditions of Theorem 3 hold true.

From the assumption that \(v(y) < \infty\), \(y > 0\), and the duality relations (22) between \(u\) and \(v\), we deduce that
\[
\lim_{x \to \infty} \frac{u(x)}{x} = \lim_{x \to \infty} u'(x) = 0. \tag{26}
\]
Lemma 1 now implies that the optimal solution \(\hat{g}(x)\) to (19) exists, for any \(x > 0\). The strict concavity of \(U\) implies the uniqueness of \(\hat{g}(x)\), as well as the fact that the function \(u\) is strictly concave too. The remaining assertions of item 1 related to the function \(v\) follow from the established properties of \(u\), because of the duality relations (22) (see, e.g., [12]).
Let \( x > 0, y = u'(x), \hat{g}(x) \) and \( \hat{h}(y) \) be the optimal solutions to (19) and (20), respectively. We have

\[
\mathbb{E}[|V(\hat{h}(y)) + \hat{g}(x)\hat{h}(y) - U(\hat{g}(x))|] \\
= \mathbb{E}[V(\hat{h}(y)) + \hat{g}(x)\hat{h}(y) - U(\hat{g}(x))] \\
\leq v(y) + xy - u(x) = 0,
\]

where, in the last step, we have used the relation \( y = u'(x) \). It follows that

\[ U(\hat{g}(x)) = V(\hat{h}(y)) + \hat{g}(x)\hat{h}(y). \]

This readily implies that

\[ U'(\hat{g}(x)) = \hat{h}(y) \quad \text{a.s.} \]

and

\[ \mathbb{E}[\hat{g}(x)\hat{h}(y)] = \mathbb{E}[U(\hat{g}(x))] - \mathbb{E}[V(\hat{h}(y))] = u(x) - v(y) = xy. \]

We complete the section with Proposition 1, which was used in the proof of item 3 of Theorem 2. This proposition was proved in [10] under the additional assumption \( AE(U) < 1 \).

Let \( \tilde{D} \) be a convex subset of \( D \) such that:

1. For any \( g \in C \),

\[ \sup_{h \in \tilde{D}} \mathbb{E}[gh] = \sup_{h \in D} \mathbb{E}[gh]. \quad (27) \]

2. The set \( \tilde{D} \) is closed under countable convex combinations, that is, for any sequence \((h^n)_{n \geq 1}\) of elements of \( \tilde{D} \) and any sequence of positive numbers \((a^n)_{n \geq 1}\) such that \( \sum_{n=1}^{\infty} a^n = 1 \) the random variable \( \sum_{n=1}^{\infty} a^n h^n \) belongs to \( \tilde{D} \).

**Proposition 1.** Assume that the conditions of Theorem 4 hold true and that \( \tilde{D} \) satisfies the above assertions. The value function \( v(y) \) defined in (20) then satisfies

\[ v(y) = \inf_{h \in \tilde{D}} \mathbb{E}[V(yh)]. \quad (28) \]

The proof of the proposition will use the following two lemmas.

The first is an easy result, whose proof is analogous to the proof of Proposition 3.1 in [10] and is therefore skipped.

**Lemma 2.** Under the assumptions of Proposition 1, let \( \hat{h}(y) \) be the optimal solution to (20). Then there exists a sequence \((h^n)_{n \geq 1}\) in \( \tilde{D} \), that converges almost surely to \( \hat{h}(y)/y \).
**Lemma 3.** Under the assumptions of Proposition 1, we have, for each \( y > 0 \),

\[
\inf_{h \in \mathcal{D}} \mathbb{E}[V(yh)] < \infty.
\]

**Proof.** To simplify the notation we shall prove the assertion of the lemma for the case \( y = 1 \).

Let \((\lambda_n)_{n \geq 1}\) be a sequence of strictly positive numbers such that \(\sum_{n=1}^{\infty} \lambda_n = 1\). We denote by \(\hat{h}(\lambda_n)\) the optimal solution to (20) corresponding to the case \(y = \lambda_n\).

Let \((\delta_n)_{n \geq 2}\) be a sequence of strictly positive numbers, decreasing to 0, such that

\[
\sum_{n=1}^{\infty} \mathbb{E}[V(\hat{h}(\lambda_n))I(A_n)] < \infty \quad \text{if } A_n \in \mathcal{F}, \mathbb{P}[A_n] \leq \delta_n, \; n \geq 2.
\]

From Lemma 2 we deduce the existence of a sequence \((h_n)_{n \geq 1}\) in \(\tilde{D}\) such that

\[
\mathbb{P}[V(\lambda_nh_n) > V(\hat{h}(\lambda_n)) + 1] \leq \delta_{n+1}, \; n \geq 1.
\]

We define the sequence of measurable sets \((A_n)_{n \geq 1}\) as follows:

\[
A_1 = \{V(\lambda_1h_1) \leq V(\hat{h}(\lambda_1) + 1)\}
\]

\[
\vdots
\]

\[
A_n = \{V(\lambda_nh_n) \leq V(\hat{h}(\lambda_n) + 1)\} \setminus \bigcup_{k=1}^{n-1} A_k.
\]

This sequence has the following properties:

\[
A_i \cap A_j = \emptyset \quad \text{if } i \neq j,
\]

\[
\mathbb{P}\left[\bigcup_{n=1}^{\infty} A_n\right] = 1,
\]

\[
\mathbb{P}[A_n] \leq \delta_n, \quad n \geq 2.
\]

We define

\[
h = \sum_{n=1}^{\infty} \lambda_nh_n.
\]

We have \(h \in \tilde{D}\) because the set \(\tilde{D}\) is closed under countable convex combinations.

The proof now follows from the inequalities

\[
\mathbb{E}[V(h)] = \sum_{n=1}^{\infty} \mathbb{E}[V(h)I(A_n)] \leq \sum_{n=1}^{\infty} \mathbb{E}[V(\lambda_nh_n)I(A_n)].
\]
where (i) holds true because $V$ is a decreasing function, (ii) follows from the construction of the sequence $(A_n)_{n \geq 1}$, and (iii) is a consequence of (29). □

**Proof of Proposition 1.** Fix $\varepsilon > 0$ and $y > 0$. We have to show that there is $h \in \tilde{D}$ such that

$$\mathbb{E}[V((y + \varepsilon)h)] \leq v(y) + \varepsilon. \tag{i}$$

Let $\hat{h} = \hat{h}(y)$ be the optimal solution to the optimization problem (20) and $f$ be an element of $\tilde{D}$ such that

$$\mathbb{E}[V(\varepsilon f)] < \infty. \tag{ii}$$

The existence of such a function $f$ follows from Lemma 3. Let $\delta > 0$ be a sufficiently small number such that

$$\mathbb{E}[\left\{|V(\hat{h})| + |V(\varepsilon f)|\right\} I(A)] \leq \frac{\varepsilon}{2} \quad \text{if } A \in \mathcal{F}, \quad \mathbb{P}[A] \leq \delta. \tag{30}$$

From Lemma 2 we deduce the existence of $g \in \tilde{D}$ such that

$$\mathbb{P}\left[V(yg) > V(\hat{h}) + \frac{\varepsilon}{2}\right] \leq \delta. \tag{31}$$

Denote

$$A = \left\{V(yg) > V(\hat{h}) + \frac{\varepsilon}{2}\right\}$$

and define

$$h = \frac{yg + \varepsilon f}{y + \varepsilon}.$$ 

Since the set $\tilde{D}$ is convex, $h \in \tilde{D}$. The proof now follows from the inequalities.

$$\mathbb{E}[V((y + \varepsilon)h)] = \mathbb{E}[V(yg + \varepsilon f)] \leq \mathbb{E}[V(yg)I(A^c)] + \mathbb{E}[V(\varepsilon f)I(A)] \leq v(y) + \varepsilon, \tag{ii}$$

where (i) holds true, because $V$ is a decreasing function, and (ii) follows from (30) and (31). □
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