Double Skorokhod Map and Reneging Real-Time Queues

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Abstract

An explicit formula for the Skorokhod map $\Gamma_{0,a}$ on $[0,a]$ for $a > 0$ is provided and related to similar formulas in the literature. Specifically, it is shown that on the space $D[0,\infty)$ of right-continuous functions with left limits taking values in $\mathbb{R}$,

$$\Gamma_{0,a}(\psi)(t) = \psi(t) - \left[ (\psi(0) - a)^+ \wedge \inf_{u \in [0,t]} \psi(u) \right] \vee \sup_{s \in [0,t]} \left[ (\psi(s) - a) \wedge \inf_{u \in [s,t]} \psi(u) \right].$$

is the unique function taking values in $[0,a]$ that is obtained from $\psi$ by minimal “pushing” at the endpoints 0 and $a$. An application of this result to real-time queues with reneging is outlined.
1 Introduction

In 1961 A. V. Skorokhod [12] considered the problem of constructing solutions to stochastic differential equations on the half-line $\mathbb{R}_+$ with a reflecting boundary condition at 0. His construction implicitly used properties of a deterministic mapping on the space $\mathcal{C}[0, \infty)$ of continuous functions on $[0, \infty)$. This mapping was used more explicitly by Anderson and Orey in their study of large deviations properties of reflected diffusions on a half-space in $\mathbb{R}^n$ (see p. 194 of [1]). These authors exploited the fact that the mapping, which is now called the Skorokhod map and is denoted here by $\Gamma_0$, has the explicit representation

$$\Gamma_0(\psi)(t) = \psi(t) + \max_{s \in [0,t]} [-\psi(s)]^+, \quad t \geq 0,$$

and is consequently Lipschitz continuous (with constant 2). In fact, this formula easily extends to a mapping on $D(0, \infty)$ the space of right-continuous functions with left limits mapping $[0, \infty)$ into $\mathbb{R}$. Given $\psi \in D(0, \infty)$, define

$$\eta(t) = \sup_{s \in (0,t]} [-\psi(s)]^+ = -\inf_{s \in (0,t]} [\psi(s) \wedge 0]$$

(1.1)

and

$$\Gamma_0(\psi) = \psi + \eta \quad \forall \psi \in D(0, \infty).$$

(1.2)

(Here, and in what follows, the operations $\lor$ and $\land$ denote maximum and minimum, respectively, and $x^+$ represents $x \lor 0$.) From (1.1) and (1.2) it is clear that $\Gamma_0(\psi)$ is in $D(0, \infty)$ and takes values in $\mathbb{R}_+$, $\eta$ is in $D(0, \infty)$ and is nondecreasing, and the pair of functions $(\Gamma_0(\psi), \eta)$ satisfies the complementarity condition

$$\int_0^\infty \mathbb{I}_{\{\Gamma_0(\psi)(s) > 0\}} \, d\eta(s) = 0,$$

(1.3)

which says that $\eta$ “pushes” only when $\Gamma_0(\psi)$ is zero. These properties uniquely characterize the pair of functions $(\Gamma_0(\psi), \eta)$, and this pair is said to solve the Skorokhod problem for $\psi$ on $[0, \infty)$.

Let $z < a$ be a real number. The double Skorokhod map $\Gamma_{z,a}$ is the mapping from $D(0, \infty)$ into itself such that for $\psi \in D(0, \infty)$, $\Gamma_{z,a}(\psi)$ takes values in $[z, a]$ and has the decomposition

$$\Gamma_{z,a}(\psi) = \psi + \eta_\ell - \eta_u,$$

(1.4)

where $\eta_\ell$ and $\eta_u$ are nondecreasing functions in $D(0, \infty)$ so that the triple $(\Gamma_{z,a}(\psi), \eta_\ell, \eta_u)$ satisfies the complementarity conditions

$$\int_0^\infty \mathbb{I}_{\{\Gamma_{z,a}(\psi)(s) > z\}} \, d\eta_\ell(s) = 0, \quad \int_0^\infty \mathbb{I}_{\{\Gamma_{z,a}(\psi)(s) < a\}} \, d\eta_u(s) = 0.$$

(1.5)

The function $\eta_\ell$ “pushes” only when $\Gamma_{z,a}(\psi)$ is at the lower boundary $z$, and $\eta_u$ “pushes” only when $\Gamma_{z,a}(\psi)$ is at the upper boundary $a$. Existence and uniqueness of $\eta_\ell$ and $\eta_u$, and hence the validity of the definition of $\Gamma_{z,a}(\psi)$ for continuous functions $\psi$ as well as step functions in $D(0, \infty)$, follow directly from Tanaka [13], Lemmas 2.1, 2.3 and 2.6. In fact, it is well known that for every $\psi \in D(0, \infty)$, there exist unique nondecreasing $\eta_\ell$ and $\eta_u$ in $D(0, \infty)$ so that $\Gamma_{z,a}(\psi)$ is a function in $D(0, \infty)$ taking values in $[z, a]$ and (1.5) is satisfied (see, e.g., [2], [15]). The triple $(\Gamma_{z,a}(\psi), \eta_\ell, \eta_u)$ is said to solve the Skorokhod problem for $\psi$ on $[z, a]$.

In contrast to the formulas (1.1), (1.2) for the Skorokhod map on $[0, \infty)$, a unified explicit formula for the Skorokhod map $\Gamma_{z,a}$ on the space $D(0, \infty)$ was obtained only recently. Such a
formula was provided for $\Gamma_0, a$ by [10] as a composition of maps with explicit formulas, and a
formula for $\Gamma_z, a$ with $z \neq 0$ can be immediately obtained by translation. Given $\phi \in D[0, \infty)$,
define
$$\Lambda_a(\phi)(t) = \phi(t) - \sup_{s \in [0, t]} \left( (\phi(s) - a)^+ \wedge \inf_{u \in [s, t]} \phi(u) \right). \quad (1.6)$$
It is shown in [10] that $\Gamma_0, a = \Lambda_a \circ \Gamma_0$. \quad (1.7)

The explicit formula (1.7) and (1.8) below are potentially useful in the study of the properties
of the double Skorokhod map. Indeed, continuity and monotonicity properties of this map are
developed in [10] using formula (1.7). The explicit formula and related monotonicity properties
have also been generalized to the case of a time-varying interval in [3].

It has recently come to the attention of the authors of this paper that there are several different
formulas in the literature for $\Gamma_0, a$ restricted to various subsets of $D[0, \infty)$. The earliest case of
such a formula appears to be Lemma 2 of [4], where the double reflection of a process that is the
sum of an Itô integral and an integral with respect to time is obtained. Equation (15) in Cooper,
Schmidt and Serfozo [5] provides a formula for $\Gamma_0, a$ restricted to the set of continuous, bounded-
variation functions in $D[0, \infty)$. Although the authors do not discuss its relation to the Skorokhod
map, Lemma 5.6 of Ganesh, O’Connell and Wischik [7] give a formula for $\Gamma_{0, a}$ restricted to a
subset of the continuous, bounded-variation functions. Equations (4) and (5) of Toomey [14] give
an analogue of $\Gamma_{0, a}$ for functions on the integers rather than on $[0, \infty)$. Functions defined on the
integers can be regarded as piecewise constant functions in $D[0, \infty)$. Because the formulas provided by these papers are Lipschitz continuous in the Skorokhod metric, their continuous extensions to the closures of the sets on which they are
specified must be given by the same formulas. A separate argument (see [15]) can be used to show
that $\Gamma_0, a$ is also Lipschitz continuous on $D[0, \infty)$. Therefore, these extended formulas must agree
with $\Gamma_0, a$ on their domains of definition. This suggests that all these formulas are closely related.
One purpose of this paper is to work out these relationships. In doing so, we discover another
formulation of (1.7) that avoids the need to compose mappings. In particular, in Section 2 we show that
$$\Gamma_{0, a}(\psi)(t) = \psi(t) - \left( (\psi(0) - a)^+ \wedge \inf_{u \in [0, t]} \psi(u) \right) \lor \sup_{s \in [0, t]} \left( (\psi(s) - a) \wedge \inf_{u \in [s, t]} \psi(u) \right) \quad (1.8)$$
for all $\psi \in D[0, \infty)$. (In the above formula, and throughout this article, the operations $\land$ and
$\lor$ have priority over the operations $+$ and $-$. ) The relation between this formula and the other
formulas that have appeared in the literature is discussed in Section 3.

There is a second purpose for this paper. The derivation of the explicit formula for $\Gamma_{0, a}$
developed in [10] was motivated by the analysis of real-time queues with reneging (see [11]). Because
the derivation in [11] is highly technical, in Section 4 we provide a non-rigorous but more accessible
explanation of that application.

2 Alternative formula for $\Gamma_{0, a}$

We begin with the proof of (1.8).
Theorem 2.1 For \( \psi \in \mathcal{D}[0, \infty) \), define

\[
\Xi_a(\psi)(t) = \psi(t) - \left[ (\psi(0) - a)^+ \wedge \inf_{u \in [0, t]} \psi(u) \right] \vee \sup_{s \in [0, t]} \left[ (\psi(s) - a) \wedge \inf_{u \in [s, t]} \psi(u) \right].
\] (2.1)

Then \( \Xi_a = \Lambda_a \circ \Gamma_0 = \Gamma_{0,a} \).

PROOF: We show that \( \Xi_a = \Lambda_a \circ \Gamma_0 \) and then appeal to (1.7). Let \( \psi \in \mathcal{D}[0, \infty) \) be given. We have immediately from (2.1) that

\[
\Xi_a(\psi)(0) = \psi(0) - (\psi(0) - a)^+ \wedge \psi(0) = \begin{cases} \ a & \text{if } \psi(0) \geq a, \\ \psi(0) & \text{if } 0 \leq \psi(0) \leq a, \\ 0 & \text{if } \psi(0) \leq 0. \end{cases}
\]

This agrees with \( \Lambda_a \circ \Gamma_0(0) \) since, by (1.1), (1.2) and (1.6), \( \Gamma_0(\psi)(0) = \psi(0) \vee 0 \) and \( \Lambda_a(\phi)(0) = \phi(0) \wedge a \) when \( \phi(0) \geq 0 \).

Now let \( t > 0 \) be fixed. Let us define \( \phi = \psi + \eta \), where \( \eta \) is given by (1.1). In other words, \( \phi = \Gamma_0(\psi) \). Let us next define \( \overline{\phi} = \Lambda_a(\phi) \). We must show that \( \overline{\phi}(t) = \Xi_a(\psi)(t) \).

Case I: \( \eta(t) = 0 \).

Because \( \eta \) is nondecreasing, in this case we have \( \eta(s) = 0 \) and \( \phi(s) = \psi(s) \) for all \( s \in [0, t] \). In particular, \( \psi \) is nonnegative on \([0, t]\). Therefore, for every \( s \in [0, t] \), \( 0 \leq (\psi(s) - a)^+ \wedge \inf_{u \in [s, t]} \psi(u) \). Since the latter expression can be rewritten as \( 0 \vee \left[ (\psi(s) - a) \wedge \inf_{u \in [s, t]} \psi(u) \right] \), it follows that

\[
\Xi_a(\psi)(t) = \psi(t) - 0 \vee \sup_{s \in [0, t]} \left[ (\psi(s) - a) \wedge \inf_{u \in [s, t]} \psi(u) \right] = \psi(t) - \sup_{s \in [0, t]} \left[ (\psi(s) - a)^+ \wedge \inf_{u \in [s, t]} \psi(u) \right] = \Lambda_a(\psi)(t) = \Lambda_a(\phi)(t) = \overline{\phi}(t).
\]

Case II: \( \eta(t) > 0 \).

In this case, (1.1) becomes \( \eta(t) = -\inf_{u \in [0, t]} \psi(u) \). Using \( \psi = \phi - \eta \), we write (2.1) as

\[
\Xi_a(\psi)(t) = \phi(t) - \left\{ \eta(t) + \left[ (\phi(0) - a - \eta(0))^+ \wedge (-\eta(t)) \right] \vee \sup_{s \in [0, t]} \left[ (\phi(s) - a - \eta(s)) \wedge \inf_{u \in [s, t]} (\phi(u) - \eta(u)) \right] \right\} \\
\quad = \phi(t) - \left[ ((\phi(0) - a - \eta(0))^+ + \eta(t)) \wedge 0 \right] \vee \sup_{s \in [0, t]} \left[ \phi(s) - a + \eta(t) - \eta(s) \right] \wedge \inf_{u \in [s, t]} \phi(u) + \eta(t) - \eta(u)).
\]

The term \( (\phi(0) - a - \eta(0))^+ + \eta(t) \) is nonnegative, and so \( (\phi(0) - a - \eta(0))^+ + \eta(t)) \wedge 0 = 0 \). Therefore,

\[
\Xi_a(\psi)(t) = \phi(t) - \sup_{s \in [0, t]} \left[ (\phi(s) - a + \eta(t) - \eta(s)) \wedge \inf_{u \in [s, t]} (\phi(u) + \eta(t) - \eta(u)) \right]^+.
\]
We conclude the proof that this last expression is \( \bar{\phi}(t) \triangleq \Lambda_a(\phi)(t) \) by showing that

\[
\left( \phi(s) - a + \eta(t) - \eta(s) \right) \land \inf_{u \in [s,t]} \left( \phi(u) + \eta(t) - \eta(u) \right) = \left( \phi(s) - a \right)^+ \land \inf_{u \in [s,t]} \phi(u). \tag{2.2}
\]

There are two possibilities. The first is that \( \phi(u) \triangleq \Gamma_0(\psi)(u) > 0 \) for every \( u \in [s,t] \). According to the complementarity condition (1.3), \( \eta \) is then constant on \([s,t] \), and the left-hand side of (2.2) becomes \( \left( \phi(s) - a \right) \land \inf_{u \in [s,t]} \phi(u) \), which agrees with the right-hand side.

The other possibility is that \( \phi(u) = 0 \) for some \( u \in [s,t] \). Define \( u_* = \sup \{ u \in [s,t] : \phi(u) = 0 \} \). According to the complementarity condition (1.3), either \( \phi(u_*) = 0 \) and \( \eta \) is constant on \([u_*,t] \) or else \( u_* > s \), \( \phi(u_*) > 0 \), \( \eta \) is constant on \([u_*,t] \), and \( \eta \) is continuous at \( u_* \). In either case, \( \inf_{u \in [s,t]}(\phi(u) + \eta(t) - \eta(u)) = 0 \) and \( \inf_{u \in [s,t]} \phi(u) = 0 \), and hence (2.2) holds with both sides equal to zero.

\[\square\]

Remark 2.2 If \( \psi(0) \leq 0 \), then

\[
(\psi(0) - a)^+ \land \inf_{u \in [0,t]} \psi(u) = \inf_{u \in [0,t]} \psi(u),
\]

and

\[
\Xi_a(\psi)(t) = \psi(t) - \inf_{u \in [0,t]} \psi(u) \lor \sup_{s \in [0,t]} \left( (\psi(s) - a) \land \inf_{u \in [s,t]} \psi(u) \right) = \psi(t) - \sup_{s \in [0,t]} \left( (\psi(s) - a) \land \inf_{u \in [s,t]} \psi(u) \right) \land \inf_{u \in [0,t]} \psi(u) = \psi(t) - \sup_{s \in [0,t]} \left( (\psi(s) - a) \land \inf_{u \in [s,t]} \psi(u) \right) \land \inf_{u \in [0,t]} \psi(u). \tag{2.3}
\]

Example 2.3 We provide here an example illustrating the fact that \( \Xi_a = \Lambda_a \circ \Gamma_0 \), and demonstrating in addition that \( \Xi_a \neq \Lambda_a \). Let \( a = 1 \) and

\[
\psi(t) = \begin{cases} 
-2 + t, & 0 \leq t \leq 4, \\
6 - t, & 4 \leq t \leq 6.
\end{cases} \tag{2.4}
\]

For \( 0 \leq t \leq 6 \), we have \( \inf_{u \in [0,t]} \psi(u) = \psi(0) = -2 \). It is straightforward to compute

\[
\sup_{s \in [0,t]} \left( (\psi(s) - 1) \lor (-2) \right) \land \inf_{u \in [s,t]} \psi(u) = \begin{cases} 
-2, & 0 \leq t \leq 1, \\
-3 + t, & 1 \leq t \leq 4, \\
1, & 4 \leq t \leq 5, \\
6 - t, & 5 \leq t \leq 6.
\end{cases}
\]

According to (2.3),

\[
\Xi_a(\psi)(t) = \psi(t) - \sup_{s \in [0,t]} \left( (\psi(s) - 1) \lor (-2) \right) \land \inf_{u \in [s,t]} \psi(u) = \begin{cases} 
t, & 0 \leq t \leq 1, \\
1, & 1 \leq t \leq 4, \\
5 - t, & 4 \leq t \leq 5, \\
0, & 5 \leq t \leq 6.
\end{cases}
\]
We see that $\Xi_a(\psi) \neq \Lambda_a(\psi)$ because

$$
\sup_{s \in [0,t]} \left[ (\psi(s) - 1)^+ \wedge \inf_{u \in [s,t]} \psi(u) \right] = \begin{cases} 
-2 + t, & 0 \leq t \leq 2, \\
0, & 2 < t \leq 3, \\
-3 + t, & 3 \leq t \leq 4, \\
1, & 4 < t \leq 5, \\
6 - t, & 5 < t \leq 6,
\end{cases}
$$

and so

$$
\Lambda_a(\psi)(t) = \psi(t) - \sup_{s \in [0,t]} \left[ (\psi(s) - 1)^+ \wedge \inf_{u \in [s,t]} \psi(u) \right] = \begin{cases} 
0, & 0 \leq t \leq 2, \\
-2 + t, & 2 < t \leq 3, \\
1, & 3 \leq t \leq 4, \\
5 - t, & 4 < t \leq 5, \\
0, & 5 < t \leq 6.
\end{cases}
$$

The discrepancy between $\Xi_a(\psi)$ and $\Lambda_a(\psi)$ is due to the fact that $\psi$ can take negative values. If $\psi$ is nonnegative, then $\Gamma_0(\psi) = \psi$, and Theorem 2.1 implies $\Xi_a(\psi) = \Lambda_a \circ \Gamma_0(\psi) = \Lambda_a(\psi)$. For $\psi$ given by (2.4),

$$
\phi(t) = \Gamma_0(\psi)(t) = \psi(t) + 2 = \begin{cases} 
t, & 0 \leq t \leq 4, \\
8 - t, & 4 < t \leq 6,
\end{cases}
$$

and

$$
\sup_{s \in [0,t]} \left[ (\phi(s) - 1)^+ \wedge \inf_{u \in [s,t]} \phi(u) \right] = \begin{cases} 
0, & 0 \leq t \leq 1, \\
-1 + t, & 1 < t \leq 4, \\
3, & 4 \leq t \leq 5, \\
8 - t, & 5 < t \leq 6.
\end{cases}
$$

Therefore,

$$
\Lambda_a(\phi)(t) = \phi(t) - \sup_{s \in [0,t]} \left[ (\phi(s) - 1)^+ \wedge \inf_{u \in [s,t]} \phi(u) \right] = \begin{cases} 
t, & 0 \leq t \leq 1, \\
1, & 1 < t \leq 4, \\
5 - t, & 4 < t \leq 5, \\
0, & 5 < t \leq 6.
\end{cases}
$$

This illustrates the result $\Xi_a(\psi) = \Lambda_a \circ \Gamma_0(\psi) = \Lambda_a(\phi)$ of Theorem 2.1. \hfill \Box

### 3 Comparison with Other Formulas

#### 3.1 The formula of Cooper, Schmidt and Serfozo [5]

Following Cooper, Schmidt and Serfozo [5], we let $H$ be a signed measure on the Borel subsets of $\mathbb{R}_+$ whose total variation on each compact interval is finite. The function $t \mapsto H(0,t]$ is right-continuous with left limits and is of bounded variation. We denote this function by $H(0,\cdot]$. Let $a$ be a positive number, and let $x \in [-a,0]$ be given. Cooper et. al. [5] (equation [15]) define

$$
X(t) \overset{\Delta}{=} \sup_{s \in [0,t]} \inf_{u \in [s,t]} \left[ x \mathbb{1}_{\{s-u=0\}} + H(u,t] - a \mathbb{1}_{\{s-u>0\}} \right]
$$

and show that $X = \Gamma_{-a,0}(x + H(0,\cdot])$. In particular, $X(0) = x$. 

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Negating (3.1), we obtain
\[-X(t) = - \sup_{s \in [0,t]} \inf_{u \in [s,t]} \left[ x^I_{\{s=u=0\}} + H(u,t) - aI_{\{s=u>0\}} \right], \tag{3.2}\]
and the result in [5] implies that
\[-X = \Gamma_{0,a} \left( -x - H(0, \cdot) \right). \tag{3.3}\]

In particular, \(-X(0) = -x\).

To relate (3.2) to \(\Xi_a\), we let \(\psi\) be a bounded variation function in \(D[0, \infty)\). We can then define the signed measure \(H\) by
\[H(u,t) = \psi(u) - \psi(t), \quad 0 \leq u \leq t. \tag{3.4}\]
The number \(-x\) in (3.3) must be taken to be in the interval \([0,a]\). We define \(-x\) in terms of \(\psi\) by
\[-x = \Gamma_{0,a}(\psi)(0) = [\psi(0)]^+ \land a. \tag{3.5}\]
It is then easily verified that
\[X + \psi(0) = (\psi(0) - a)^+ \land \psi(0). \tag{3.6}\]

With the choices of \(H\) and \(-x\) given by (3.4) and (3.5), (3.2) becomes
\[-X(t) = - \sup_{s \in [0,t]} \inf_{u \in [s,t]} \left[ x^I_{\{s=u=0\}} + \psi(u) - \psi(t) - aI_{\{s=u>0\}} \right] \tag{3.3}\]
\[= \psi(t) - \inf_{u \in [0,t]} \left[ x^I_{\{u=0\}} + \psi(u) \right] \lor \sup_{s \in (0,t]} \inf_{u \in [s,t]} \left[ \psi(u) - aI_{\{s=u\}} \right] \tag{3.7}\]
\[= \psi(t) - \left[ (x + \psi(0)) \land \inf_{u \in [0,t]} \psi(u) \right] \lor \sup_{s \in (0,t]} \left[ (\psi(s) - a) \land \inf_{u \in [s,t]} \psi(u) \right] \tag{3.8}\]
Because \(\psi\) is right continuous, for \(0 \leq s \leq t\),
\[\left( \psi(s) - a \right) \land \inf_{u \in [s,t]} \psi(u) = \left( \psi(s) - a \right) \land \inf_{u \in [s,t]} \psi(u), \tag{3.9}\]
where we adopt the convention that \(\inf_{u \in (t,t]} \psi(u) = \infty\) to handle the case \(s = t\). Furthermore, this expression is right-continuous in \(s\). Therefore, for \(t > 0\),
\[\sup_{s \in (0,t]} \left[ (\psi(s) - a) \land \inf_{u \in (s,t]} \psi(u) \right] = \sup_{s \in [0,t]} \left[ (\psi(s) - a) \land \inf_{u \in [s,t]} \psi(u) \right]. \tag{3.10}\]
We thus conclude from (3.7) that
\[-X(t) = \psi(t) - \left[ (\psi(0) - a)^+ \land \inf_{u \in [0,t]} \psi(u) \right] \lor \sup_{s \in [0,t]} \left[ (\psi(s) - a) \land \inf_{u \in [s,t]} \psi(u) \right] = \Xi_a(\psi)(t). \tag{3.11}\]

To deal with the case that \(t = 0\), it is easily verified that
\[\Xi_a(\psi)(0) = \psi(0) - (\psi(0) - a)^+ \land \psi(0) = [\psi(0)]^+ \land a = \Gamma_{0,a}(\psi)(0) = \Lambda_a \circ \Gamma_0(\psi)(0) = -X(0). \tag{3.12}\]
\[\square\]
3.2 The formula of Chitashvili and Lazrieva [4]

Lemma 2 of [4] considers the process
\[ \xi(t) = \xi(0) + \int_0^t a(s) \, ds + \int_0^t b(s) \, dW(s), \]  
(3.10)
where \( W \) is a Brownian motion, \( a(\cdot) \) and \( b(\cdot) \) are random processes adapted to the filtration generated by \( W \), and \( \xi(0) \in [\gamma_1, \gamma_2] \). Here \( \gamma_1 < \gamma_2 \) are real numbers. In Lemma 2 of [4], the doubly reflected version \( \xi^* \) of \( \xi \) on \([\gamma_1, \gamma_2]\) is shown to be
\[ \xi^*(t) = \sup_{s \in [0, t]} \inf_{u \in [0, t]} \left[ (\xi(t) - \xi(s) + \gamma_1) \mathbb{1}_{\{s > u\}} + (\xi(t) - \xi(u) + \gamma_2) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right]. \]
(3.11)

The proof of Lemma 2 in [4] is probabilistic, using the specific form (3.10) of \( \xi \).

Because the three indicators \( \mathbb{1}_{\{s > u\}} \), \( \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} \) and \( \mathbb{1}_{\{s = u = 0\}} \) appearing on the right-hand side of (3.11) sum to 1, the addition of the same constant to \( \xi \), \( \gamma_1 \) and \( \gamma_2 \) results in the addition of this constant to the right-hand side of (3.11), and hence to \( \xi^* \). We may therefore assume without loss of generality that \( \gamma_1 = -a < 0 \), \( \gamma_2 = 0 \) and \( \xi(0) \in [-a, 0] \). With this simplification, we show below that \( -\xi^* = \Xi(a, -\xi) \). Actually, all we need to show is that
\[ \xi^*(t) = \sup_{s \in [0, t]} \inf_{u \in [s, t]} \left[ (\xi(t) - \xi(s) - a) \mathbb{1}_{\{s > u\}} + (\xi(t) - \xi(u)) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right], \]
(3.12)
because we can then set \( \psi = -\xi \) and proceed as in the previous section.

We thus begin with (3.11), where \( \gamma_1 \) and \( \gamma_2 \) have been set to \(-a\) and 0 respectively, and \( \xi(0) \in [-a, 0] \). We have
\[ \xi^*(t) = \sup_{s \in [0, t]} \inf_{u \in [0, t]} \left[ (\xi(t) - \xi(s) - a) \mathbb{1}_{\{s > u\}} + (\xi(t) - \xi(u)) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right] \]
\[ = \sup_{s \in [0, t]} \left[ \inf_{u \in [s, t]} (\xi(t) - \xi(s) - a) \mathbb{1}_{\{s > u\}} + (\xi(t) - \xi(u)) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right] \]
\[ = \sup_{s \in [0, t]} \left[ \mathbb{1}_{\{s > 0\}} (\xi(t) - \xi(s) - a) + \mathbb{1}_{\{s = 0\}} \right] \wedge \inf_{u \in [s, t]} \left[ (\xi(t) - \xi(u)) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right]. \]
(3.13)

To evaluate the the right-hand side of (3.13), we consider separately the cases \( s = 0 \) and \( 0 < s \leq t \).

If \( s = 0 \), then
\[ \left[ \mathbb{1}_{\{s > 0\}} (\xi(t) - \xi(s) - a) + \mathbb{1}_{\{s = 0\}} \right] \wedge \inf_{u \in [s, t]} \left[ (\xi(t) - \xi(u)) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right] \]
\[ = \inf_{u \in [s, t]} \left[ (\xi(t) - \xi(u)) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right]. \]
(3.14)

On the other hand, if \( s > 0 \), then
\[ \left[ \mathbb{1}_{\{s > 0\}} (\xi(t) - \xi(s) - a) + \mathbb{1}_{\{s = 0\}} \right] \wedge \inf_{u \in [s, t]} \left[ (\xi(t) - \xi(u)) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right] \]
\[ = \inf_{u \in [s, t]} \left[ (\xi(t) - \xi(u)) \mathbb{1}_{\{s < u\} \cup \{s = u > 0\}} + \xi(t) \mathbb{1}_{\{s = u = 0\}} \right]. \]
(3.15)

Substituting (3.14) and (3.15) into (3.13), we obtain (3.12).
3.3 The formula of Ganesh, O’Connell and Wischik [7]

Section 5.7 of [7] records the size of a finite-buffer fluid queue at time zero under the assumption that the queue was empty at time \( t \), where \( t > 0 \). The buffer size of the queue is \( a \), a positive number. We adjust the formula in [7] by relabeling time; our queue is empty at time zero and we record its size at time \( t \). We call the netput process \( \psi \). The queue length is then \( \Gamma_{0,a}(\psi) \).

We thus begin with a bounded-variation function \( \psi \in \mathcal{D}(0, \infty) \) satisfying \( \psi(0) = 0 \). Following [7], we define

\[
M(s,t) \triangleq \inf_{u \in [s,t]} (\psi(t) - \psi(u)) = \psi(t) - \sup_{u \in [s,t]} \psi(u),
\]

\[
N(s,t) \triangleq \sup_{u \in [s,t]} (\psi(t) - \psi(u)) = \psi(t) - \inf_{u \in [s,t]} \psi(u).
\]

We note that like \( \psi \) itself, \( M(s,t) \) and \( N(s,t) \) are right-continuous with left-hand limits in \( s \). Here and elsewhere, we adopt the notational conventions

\[
\sup_{u \in [s-\varepsilon,t]} \psi(u) \triangleq \lim_{\varepsilon \to 0^+} \sup_{u \in [s-\varepsilon,t]} \psi(u), \quad \inf_{u \in [s-\varepsilon,t]} \psi(u) \triangleq \lim_{\varepsilon \to 0^+} \inf_{u \in [s-\varepsilon,t]} \psi(u). \tag{3.16}
\]

We shall also use the notation \( (s-\varepsilon,t) \triangleq [s,t] \) and \( 0^- \triangleq 0 \).

**Lemma 3.1** For all \( t \geq 0 \),

\[
\sup_{s \in [0,t]} \left[ N(s,t) \wedge (M(s,t) + a) \right] \leq \inf_{s \in [0,t]} \left[ N(s,t) \vee (M(s,t) + a) \right]. \tag{3.17}
\]

**Proof:** For \( t \geq 0 \),

\[
\sup_{s \in [0,t]} \left[ N(s,t) \wedge (M(s,t) + a) \right] = \psi(t) + \sup_{s \in [0,t]} \left[ -\inf_{u \in [s,t]} \psi(u) \wedge (a - \sup_{u \in [s,t]} \psi(u)) \right]
\]

\[
= \psi(t) - \inf_{s \in [0,t]} \left[ \inf_{u \in [s,t]} \psi(u) \vee \sup_{u \in [s,t]} (\psi(u) - a) \right]. \tag{3.18}
\]

The infimum over \( s \in [0,t] \) in the last line of (3.18) is either attained by some \( s_1 \in [0,t] \), or else there is an \( s_1 \) in \([0,t]\) for which the infimum is (in the notation (3.16))

\[
\inf_{s \in [s_1-\varepsilon,t]} \psi(u) \vee \sup_{u \in [s_1-\varepsilon,t]} (\psi(u) - a).
\]

In the former case, we let \( s'_1 \) denote \( s_1 \); in the latter case, \( s'_1 \) denotes \( s_1^- \). Capturing both cases, we say that \( s'_1 \in [0,t] \) satisfies

\[
\inf_{u \in [s'_1,t]} \psi(u) \vee \sup_{u \in [s'_1,t]} (\psi(u) - a) = \inf_{u \in [0,t]} \psi(u) \vee \sup_{u \in [s'_1,t]} (\psi(u) - a). \tag{3.19}
\]

Continuing in this way, we observe that if \( s'_1 = s_1 \), then there is an \( s_* \in [s_1,t] \) that attains \( \inf_{u \in [s_1,t]} \psi(u) \) or else there is an \( s_* \in (s_1,t) \) for which the infimum is \( \psi(s_*) \). If \( s'_1 = s_1^- \), then either there is an \( s_* \in [s_1,t] \) that attains \( \inf_{s \in [s_1,t]} \psi(u) \) or else there is an \( s_* \in (s_1,t) \) for which the infimum is \( \psi(s_*) \). If \( \psi(s_*) = \inf_{u \in [s'_1,t]} \psi(u) \), we set \( s'_* = s_* \); if \( \psi(s_*) = \inf_{u \in [s'_1,t]} \psi(u) \), then...
denotes \( s_* \). Capturing all these cases, we say that \( s'_* \subseteq [s', t] \) satisfies \( \psi(s'_*) = \inf_{u \in [s'_*, t]} \psi(u) \).

With these conventions, we have

\[
\inf_{s \in [0, t]} \left[ \inf_{u \in [s, t]} \psi(u) \lor \sup_{u \in [s, t]} (\psi(u) - a) \right] = \inf_{s \in [0, t]} \left[ \psi(u) \lor \sup_{u \in [s, t]} (\psi(u) - a) \right] \geq \psi(s'_*) \lor \sup_{u \in [s'_*, t]} (\psi(u) - a) \geq \inf_{s \in [0, t]} \left[ \psi(s) \lor \sup_{u \in [s, t]} (\psi(u) - a) \right].
\]

The reverse inequality

\[
\inf_{s \in [0, t]} \left[ \inf_{u \in [s, t]} \psi(u) \lor \sup_{u \in [s, t]} (\psi(u) - a) \right] \leq \inf_{s \in [0, t]} \left[ \psi(s) \lor \sup_{u \in [s, t]} (\psi(u) - a) \right]
\]

obviously holds. Returning to (3.18), we see that

\[
\sup_{s \in [0, t]} \left[ N(s, t) \land (M(s, t) + a) \right] = \psi(t) - \inf_{s \in [0, t]} \left[ \psi(s) \lor \sup_{u \in [s, t]} (\psi(u) - a) \right]. \tag{3.19}
\]

An analogous argument shows that

\[
\inf_{s \in [0, t]} \left[ N(s, t) \lor (M(s, t) + a) \right] = \psi(t) - \sup_{s \in [0, t]} \left[ (\psi(s) - a) \land \inf_{u \in [s, t]} \psi(u) \right]. \tag{3.20}
\]

Now choose \( s_3 \in [0, t] \), where \( s_3 \) attains the infimum on the right-hand side of (3.19) (in which case we write \( s'_3 = s_3 \)) or if no such \( s_3 \) exists, then choose \( s_3 \in (0, t] \), where \( s_3^- \) attains the infimum on the right-hand side of (3.19) (in which case we write \( s'_3 = s_3^- \)). Let \( s'_4 \) be defined analogously in connection with the supremum on the right-hand side of (3.20). If \( s'_3 \leq s'_4 \) (this means either that \( s_3 < s_4 \) or else that \( s_3 = s_4 \) and it is not the case that \( s'_3 = s_3, s'_4 = s_4^- \)), we have \( \sup_{u \in [s'_4, t]} (\psi(u) - a) \geq \psi(s'_4) - a \), and so

\[
\inf_{s \in [0, t]} \left[ \psi(s) \lor \sup_{u \in [s, t]} (\psi(u) - a) \right] = \psi(s'_4) \lor \sup_{u \in [s'_4, t]} (\psi(u) - a) \geq (\psi(s'_4) - a) \land \inf_{u \in [s'_4, t]} \psi(u) \geq \sup_{s \in [0, t]} \left[ (\psi(s) - a) \land \inf_{u \in [s, t]} \psi(u) \right]. \tag{3.21}
\]

Relation (3.17) follows from (3.19) and (3.20). On the other hand, if \( s'_3 \geq s'_4 \), then \( \psi(s'_3) \geq \inf_{u \in [s'_4, t]} \psi(u) \), and again relation (3.21) and hence relation (3.17) hold. \( \square \)

For \( x \in \mathbb{R} \) and \( \alpha \leq \beta \), define \( [x]_\alpha = (x \lor \alpha) \land \beta \). On a subset of bounded-variation functions in \( D[0, \infty) \) whose initial condition is zero, in [7] a mapping \( \Phi_\alpha \) is defined by the formula

\[
\Phi_\alpha(\psi)(t) = \Delta_{\inf_{s \in [0, t]} [N(s, t) \lor (M(s, t) + a)]} \psi(t)_{\sup_{s \in [0, t]} [N(s, t) \land (M(s, t) + a)]}, \quad \tag{3.22}
\]

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According to this definition and relations (3.19) and (3.20),
\[ \Phi_a(\psi)(t) = \left( \psi(t) \lor \sup_{s \in [0,t]} \left[ N(s,t) \land (M(s,t) \land a) \right] \right) \land \inf_{s \in [0,t]} \left[ N(s,t) \lor (M(s,t) + a) \right] \]
\[ = \left\{ \psi(t) \lor \left( \psi(t) - \inf_{s \in [0,t]} \left( \psi(s) \lor \sup_{u \in [s,t]} \left( \psi(u) - a \right) \right) \right) \right\} \]
\[ \land \left( \psi(t) - \sup_{s \in [0,t]} \left( \psi(s) - a \right) \land \inf_{u \in [s,t]} \psi(u) \right) \]
\[ = \psi(t) - \left( 0 \land \inf_{s \in [0,t]} \left( \psi(s) \lor \sup_{u \in [s,t]} \left( \psi(u) - a \right) \right) \right) \lor \sup_{s \in [0,t]} \left( \psi(s) - a \right) \land \inf_{u \in [s,t]} \psi(u) \right). \] (3.23)

**Theorem 3.2** Let \( \psi \in D[0, \infty) \) satisfy \( \psi(0) = 0 \). Then \( \Phi_a(\psi) = \Xi_a(\psi) \), where \( \Xi_a(\psi) \) is given by (2.1).

**Proof:** According to (3.23),
\[ \psi(t) - \Phi_a(\psi)(t) = \left( 0 \land A(t) \right) \lor B(t), \]
where
\[ A(t) \overset{\Delta}{=} \inf_{s \in [0,t]} \left[ \psi(s) \lor \sup_{u \in [s,t]} \left( \psi(u) - a \right) \right], \quad B(t) \overset{\Delta}{=} \sup_{s \in [0,t]} \left( \psi(s) - a \right) \land \inf_{u \in [s,t]} \psi(u) \right). \]

Using \( \psi(0) = 0 \), we obtain from equation (2.1) that
\[ \psi(t) - \Xi_a(\psi)(t) = C(t) \lor B(t), \]
where
\[ C(t) \overset{\Delta}{=} \inf_{s \in [0,t]} \psi(s). \]

To prove the theorem, we must show that
\[ \left( 0 \land A(t) \right) \lor B(t) = C(t) \lor B(t). \] (3.24)

Clearly, \( A(t) \geq C(t) \), and because \( \psi(0) = 0 \), we have also \( 0 \geq C(t) \). It follows that \( \left( 0 \land A(t) \right) \geq C(t) \), and thus
\[ \left( 0 \land A(t) \right) \lor B(t) \geq C(t) \lor B(t). \] (3.25)

If \( A(t) = C(t) \), then \( 0 \land A(t) = C(t) \) and so equality holds in (3.25). To complete the proof, we need only establish the implication
\[ A(t) > C(t) \quad \Rightarrow \quad A(t) \leq B(t). \] (3.26)

Indeed, if \( A(t) > C(t) \), (3.26) will imply \( \left( 0 \land A(t) \right) \lor B(t) \leq B(t) \leq C(t) \lor B(t) \), and we have the reverse of (3.25).

Assume
\[ A(t) > C(t). \] (3.27)
Using the notation developed for Lemma 3.1, we choose \( s_1 \in [0, t] \) so that \( \psi(s_1) = C(t) \) or \( s_1 \in (0, t] \) so that \( \psi(s_1) = C(t) \). We use \( s'_1 \) to denote \( s_1 \) in the former case and \( s_1^- \) in the latter case. We capture both cases by the relation
\[
\psi(s'_1) = C(t) \leq \psi(s) \quad \forall s \in [0, t].
\] (3.28)

We next define
\[
s_2 \triangleq \sup \left\{ s \in [0, t] \left| \left[ \psi(s) \lor \sup_{u \in [s, t]} (\psi(u) - a) \right] \land \left[ \psi(s^-) \lor \sup_{u \in [s^-, t]} (\psi(u) - a) \right] = A(t) \right\}.
\]
Either \( s_2 \in [0, t] \) and \( \psi(s_2) \lor \sup_{u \in [s_2, t]} (\psi(u) - a) = A(t) \), in which case we denote \( s_2 \) by \( s'_2 \), or else \( s_2 \in (0, t] \), \( \psi(s_2) \lor \sup_{u \in [s_2, t]} (\psi(u) - a) > A(t) \), and \( \psi(s_2^-) \lor \sup_{u \in [s_2^-, t]} (\psi(u) - a) = A(t) \), in which case we denote \( s_2^- \) by \( s'_2 \). We capture both cases by the equation
\[
\psi(s'_2) \lor \sup_{u \in [s'_2, t]} (\psi(u) - a) = A(t).
\] (3.29)

We cannot have \( s'_2 < s'_1 \) (which means \( s_2 < s_1 \) or \( s_2 = s_1, s'_2 = s_2, s'_1 = s_1 \)), for then we would have, using (3.28) and the definition of \( A(t) \),
\[
A(t) \leq \psi(s'_1) \lor \sup_{u \in [s'_1, t]} (\psi(u) - a) \leq \psi(s'_2) \lor \sup_{u \in [s'_2, t]} (\psi(u) - a) = A(t),
\]
a contradiction to the maximality property of \( s'_2 \). Therefore, \( s'_2 \geq s'_1 \).

We must have
\[
\psi(s) \geq \psi(s'_2) \quad \forall s \in [s_2, t].
\] (3.30)

If this were not the case, then we would have \( \psi(s) < \psi(s'_2) \) for some \( s \in (s_2, t] \), and then
\[
A(t) \leq \psi(s) \lor \sup_{u \in [s, t]} (\psi(u) - a) \leq \psi(s'_2) \lor \sup_{u \in [s'_2, t]} (\psi(u) - a) = A(t),
\]
which also contradicts the maximality property of \( s'_2 \).

**Case I: \( A(t) = \sup_{u \in [s'_2, t]} (\psi(u) - a) \geq \psi(s'_2) \).**

Define
\[
u_2 \triangleq \sup \left\{ u \in [s_2, t] \left| (\psi(u) - a) \lor (\psi(u) - a) = A(t) \right\}.
\]

Then either \( \psi(u_2) - a = A(t) \) or else \( u_2 > 0 \), \( \psi(u_2) - a < A(t) \), \( \psi(u_2^-) - a = A(t) \). Let us consider first the case that \( \psi(u_2) - a = A(t) \), in which case we denote \( u_2 \) by \( u'_2 \). There cannot exist \( u_3 \in (u_2, t] \) for which \( \psi(u_3) < A(t) \), for if such a \( u_3 \) were to exist, we would have \( \psi(u_3) \lor \sup_{u \in [u_3, t]} (\psi(u) - a) < A(t) \). Therefore,
\[
\psi(u) \geq A(t) \quad \forall u \in [u_2, t].
\] (3.31)

Let us next consider the case that \( u_2 > 0 \), \( \psi(u_2) - a < A(t) \) and \( \psi(u_2^-) - a = A(t) \), in which case we denote \( u_2^- \) by \( u'_2 \). There cannot exist \( u_3 \in [u_2, t] \) such that \( \psi(u_3) < A(t) \), for if such a \( u_3 \) were to exist, we would again have \( \psi(u_3) \lor \sup_{u \in [u_3, t]} (\psi(u) - a) < A(t) \). Once again, (3.31) holds. From (3.31) and the fact that \( \psi(u'_2) - a = A(t) \), we have immediately
\[
B(t) \geq (\psi(u'_2) - a) \land \inf_{u \in [u'_2, t]} \psi(u) = A(t).
\]
This completes the proof of (3.26) in Case I.

**Case II:** \( A(t) = \psi(s'_1) > \sup_{u \in [s'_1, t]} (\psi(u) - a) \).

Define
\[
    u_1 \triangleq \sup \{ u \in [s_1, s_2] \mid (\psi(u) - a) \vee (\psi(u) - a) \geq \psi(s'_2) \}.
\]

If no such \( u_1 \) were to exist, then we would have \((\psi(u) - a) \vee (\psi(u) - a) < \psi(s'_2)\) for all \( u \in [s_1, s_2] \), in which case we would have from (3.27), (3.28), and the case assumption that
\[
    \psi(s'_1) \vee \sup_{u \in [s'_1, t]} (\psi(u) - a) = \psi(s'_1) \vee \sup_{u \in [s'_1, s_2]} (\psi(u) - a) \vee \sup_{u \in [s_2, t]} (\psi(u) - a) < \psi(s'_2).
\]

But according to the definition of \( A(t) \), it is dominated by the left-hand side of this expression. We have a contradiction to the case assumption, which shows that \( u_1 \in [s_1, s_2] \) is well defined.

If \( \psi(u_1) - a \geq \psi(s'_2) \), we denote \( u_1 \) by \( u'_1 \). If this is not the case, then \( u_1 > 0 \), \( \psi(u_1) - a < \psi(s'_2) \), \( \psi(u_1) - a = \psi(s'_2) \), and we denote \( u_1 \) by \( u'_1 \). We capture both cases by the relation
\[
    \psi(u'_1) - a \geq \psi(s'_2) = A(t).
\]

(3.32)

The maximality property of \( u'_1 \) implies that
\[
    \sup_{u \in [u'_1, s_2]} (\psi(u) - a) < \psi(s'_2).
\]

If \( \psi(u_3) < \psi(s'_2) \) for some \( u_3 \in (u'_1, s_2] \), then we would have
\[
    A(t) \leq \psi(u_3) \vee \sup_{u \in [u_3, t]} (\psi(u) - a) < \psi(s'_2),
\]

a contradiction to the case assumption. Therefore,
\[
    \psi(u) \geq \psi(s'_2) \quad \forall u \in (u'_1, s_2],
\]

(3.33)

It follows that
\[
    B(t) \geq (\psi(u'_1) - a) \wedge \inf_{u \in [u'_1, t]} \psi(u)
\]
\[
    \geq (\psi(u'_1) - a) \wedge \psi(u'_1) \wedge \inf_{u \in [u'_1, s_2]} \psi(u) \wedge \inf_{u \in [s_2, t]} \psi(u),
\]

and each of these terms dominates \( \psi(s'_2) = A(t) \) by (3.32), (3.33), and (3.30). This completes the proof of (3.26) in Case II, and thus completes the proof of Theorem 3.2. \( \square \)

### 3.4 The formula of Toomey [14]

Toomey [14] records the size of a finite-buffer queue at time \(-k\) under the assumption that the queue was of size \( q_m \) at time \(-m \leq -k\). The buffer size of the queue is \( a \), a positive number, and \( q_m \) is assumed to be in \([0, a] \). There are two formulas, (4) and (5), in [14], each obtained from the other by reversing the spatial axis. We deal with (5), mapping \(-m\) into time zero and mapping \(-k\) into time \( t > 0 \) and writing the formula for piecewise constant functions in \( D(0, \infty) \) rather than functions defined on the integers.
The netput process, cumulative arrivals minus offered service, over the time interval \(-k\) to \(-m\) is denoted \(U_{km}\) by [14] and by \(\psi(t) - \psi(0)\) here. We take \(\psi(0) = q_m \in [0, a]\), the initial queue length. In our notation, formula (5) in [14] is

\[
\inf_{s \in [0,t]} \sup_{u \in [s,t]} \left[ (a + \psi(t) - \psi(s)) \lor (\psi(t) - \psi(u)) \right] \\
\land \sup_{u \in [0,t]} \left[ (q_m + \psi(t) - \psi(0)) \lor (\psi(t) - \psi(u)) \right] \\
= \psi(t) - \sup_{s \in [0,t]} \inf_{u \in [s,t]} \left[ (\psi(s) - a) \land \psi(u) \right] \lor \inf_{u \in [0,t]} [0 \lor \psi(u)].
\]

Because \(\psi\) is right-continuous and \((\psi(0) - a)^+ = 0\), this expression can be rewritten as

\[
\psi(t) - \sup_{s \in [0,t]} \left[ (\psi(s) - a) \land \inf_{u \in [s,t]} \psi(u) \right] \lor \left[ (\psi(0) - a)^+ \land \inf_{u \in [0,t]} \psi(u) \right],
\]

which is \(\Xi_a(\psi)(t)\) given by (2.1).

### 4 Application to real-time queues with reneging

#### 4.1 Heavy-traffic convergence

Consider a sequence of single station queueing systems indexed by the positive integers. In the \(n\)-th system, the interarrival times are a sequence of positive, independent, identically distributed random variables \(u_1^{(n)}, u_2^{(n)}, \ldots\) and the service times are likewise a sequence of positive, independent, identically distributed random variables \(v_1^{(n)}, v_2^{(n)}, \ldots\). The arrival rate in the \(n\)-th system is \(\lambda^{(n)} = 1/Eu_1^{(n)}\), the service rate is \(\mu^{(n)} = 1/Ev_1^{(n)}\), and the traffic intensity is \(\rho^{(n)} = \lambda^{(n)}/\mu^{(n)}\).

We assume that \(\lambda^{(n)}\) has a positive limit \(\lambda\) as \(n \to \infty\), \(\mu^{(n)}\) also has a positive limit \(\mu\) as \(n \to \infty\), \(u_1^{(n)}\) has a limiting positive variance \(\alpha^2\) as \(n \to \infty\), and \(v_1^{(n)}\) has a limiting variance \(\beta^2\) as \(n \to \infty\).

We make the heavy traffic assumption

\[
\rho^{(n)} = 1 - \frac{\gamma}{\sqrt{n}}
\]

for some nonzero constant \(\gamma\). This implies \(\lambda = \mu\).

For the \(n\)-th system, the customer arrival times are

\[
S_k^{(n)} \triangleq k \sum_{i=1}^{k} u_i^{(n)}
\]

and the customer arrival process is

\[
A^{(n)}(t) \triangleq \max \left\{ k \mid S_k^{(n)} \leq t \right\}.
\]

The work arrival process is

\[
V^{(n)}(k) \triangleq \sum_{j=1}^{k} v_j^{(n)}.
\]
The netput process
\[ N^{(n)}(t) \overset{\Delta}{=} V^{(n)}(A^{(n)}(t)) - t \]
represents the work that would be present in queue at time \( t \) if the server were never idle between times 0 and \( t \). We are taking the queue to be empty at time zero. However, the queue may be idle prior to time \( t \), and thus the work that is actually present at time \( t \) is given by the workload process
\[ W^{(n)} \overset{\Delta}{=} \Gamma_0(N^{(n)}), \quad (4.2) \]
where \( \Gamma_0 \) is defined by (1.2). The idleness process
\[ I(t) \overset{\Delta}{=} -\inf_{s \in [0,t]} N^{(n)}(s) \]
plays the role of \( \eta \) of (1.1).

The scaled workload process is
\[ \hat{W}^{(n)}(t) \overset{\Delta}{=} \frac{1}{\sqrt{n}} W^{(n)}(nt), \quad t \geq 0. \quad (4.3) \]

It is well known that the following heavy traffic convergence result holds under assumption (4.1) and the Lindeberg condition
\[
\lim_{n \to \infty} \mathbb{E} \left[ \left( v_j^{(n)} - (\lambda^{(n)})^{-1} \right)^2 I_{\{|v_j^{(n)} - (\lambda^{(n)})^{-1}| > c \sqrt{\pi}\}} \right] = 0 \quad \forall c > 0.
\]

**Theorem 4.1 (Kingman [9], Iglehart and Whitt [8])** As \( n \to \infty \),
\[ \hat{W}^{(n)} \Rightarrow W^*, \quad (4.4) \]
where \( W^* \) is a Brownian motion with drift \(-\gamma\) and variance per unit time \( \lambda(\alpha^2 + \beta^2) \), reflected at the origin so as to always be nonnegative. More precisely, define \( N^*(t) \overset{\Delta}{=} -\gamma t + \sqrt{\lambda(\alpha^2 + \beta^2)} B(t) \), where \( B \) is a standard Brownian motion. Then
\[ W^* \overset{\Delta}{=} \Gamma_0(N^*). \quad (4.5) \]

### 4.2 Lead times

In real-time queues, customer deadlines are taken into account. We introduce a sequence of lead times, which are positive, independent and identically distributed random variables \( L_1^{(n)}, L_2^{(n)}, \ldots \). Customer \( k \) arrives in system \( n \) at time \( S_k^{(n)} \) with deadline \( S_k^{(n)} + L_k^{(n)} \). The lead time of customer \( k \), which is the time until the customer’s deadline elapses, is \( L_k^{(n)} \) upon arrival of customer \( k \) and then decreases at rate one thereafter, becoming negative when the customer becomes late.

Under the heavy traffic assumption (4.1), delay in the \( n \)-th system will be of order \( \sqrt{n} \), so the lead times must also be of order \( \sqrt{n} \) to avoid trivialities. We assume therefore that there is a cumulative distribution function \( G \) independent of \( n \) such that
\[ \mathbb{P} \left\{ \frac{L_j^{(n)}}{\sqrt{n}} \leq y \right\} = G(y). \quad (4.6) \]
For technical reasons, we also assume that there exists a finite \( y^* \) for which \( G(y^*) = 1 \) and \( G(y) < 1 \) for \( y < y^* \), i.e., we cannot have a lead time in the \( n \)-th system larger than \( \sqrt{n} y^* \) but we can have lead times equal to or at least arbitrarily close to \( \sqrt{n} y^* \). We also assume that 

\[ y_* \triangleq \inf \{ y \in \mathbb{R} \mid G(y) > 0 \} \]

is strictly positive; this seems to be essential to the results we obtain.

We serve the customers using the earliest deadline first (EDF) protocol. The customer in service may be preempted by the arrival of a more urgent customer. When service eventually resumes on the preempted customer, the service begins where it left off, i.e., the work already done on that customer is not lost. We wish to determine the heavy traffic limit of the distribution of the lead times of customers in queue.

We define two measure-valued processes, \( W^{(n)} \) and \( V^{(n)} \), by specifying for every Borel subset \( B \) of \( \mathbb{R} \) that

\[
W^{(n)}(t)(B) \triangleq \left\{ \text{Work associated with customers in queue at time } t \text{ with lead times in } B \right\},
\]

\[
V^{(n)}(t)(B) \triangleq \left\{ \begin{array}{ll}
\text{Work associated with customers arrived by time } t \text{ with lead times in } B, \quad & \text{whether or not still present at time } t \\
or not still present at time } t & 
\end{array} \right\}.
\]

We define the frontier to be

\[
F^{(n)}(t) \triangleq \left\{ \begin{array}{ll}
\text{Largest lead time of any customer who has ever been in service, whether or not that customer is still present, or} \\
\sqrt{n} y^* - t & \text{if this quantity is larger than the former one}
\end{array} \right\}.
\]

We also define the scaled measure-valued workload process, scaled measure-valued work arrival process, and scaled frontier, respectively:

\[
\hat{W}^{(n)}(t)(B) \triangleq \frac{1}{\sqrt{n}} W^{(n)}(nt)(\sqrt{n} B),
\]

\[
\hat{V}^{(n)}(t)(B) \triangleq \frac{1}{\sqrt{n}} V^{(n)}(nt)(\sqrt{n} B),
\]

\[
\hat{F}^{(n)}(t) \triangleq \frac{1}{\sqrt{n}} F^{(n)}(nt).
\]

At time \( t > 0 \), work whose lead time is less or equal to \( F^{(n)}(t) \) receives priority. However, the arrival rate of work in this category is less than the full arrival rate, since \( F^{(n)}(t) < \sqrt{n} y^* \). Therefore, this work is not in heavy traffic, which leads to the following result, proved in [6].

**Lemma 4.2 (Crushing)** As \( n \to \infty \), \( \hat{V}^{(n)}(-\infty, \hat{F}^{(n)}] \to 0 \).

Lemma 4.2 says that in order to understand the limiting distribution of lead times, it is enough to consider only work whose lead time exceeds \( F^{(n)} \). However, since this work has never been in service, we can restrict attention to the measure-valued work arrival process \( V^{(n)} \) rather than the more complicated measure-valued workload process \( W^{(n)} \). The following limit for the scaled version of this process is obtained in [6].

**Theorem 4.3** For all \( y \in \mathbb{R} \),

\[
\hat{V}^{(n)}(\cdot, \infty) \Rightarrow H(y) \triangleq \int_y^\infty (1 - G(x)) \, dx.
\]
In fact, the convergence in (4.13) is weak convergence of a sequence of measure-valued processes to a measure on \( \mathbb{R} \), not just weak convergence of a real-valued process for each fixed \( y \).

Theorem 4.3 can be explained by the following heuristic. To have scaled lead time \( x \) at scaled time \( t \), a customer must have entered the system scaled time units \( s \) earlier with scaled lead time \( x + s \). Given that a customer arrives at scaled time \( t - s \), the density at time \( t \) for the lead time \( x \) at scaled time \( t \), which is therefore

\[
\lambda \int_0^\infty G'(x + s) \, ds = \lambda (G(\infty) - G(x)) = \lambda (1 - G(x)).
\]

The limiting work brought by each customer is \( 1/\mu = 1/\lambda \). Therefore, to find the density of work (as opposed to customers) with scaled lead time \( x \) at scaled time \( t \), we divide the expression above by \( \lambda \). Finally, to obtain the amount of work in \( (y, \infty) \) at time \( t \), we integrate the resulting expression from \( y \) to \( \infty \) and obtain \( H(y) \) defined in (4.13).

The limiting scaled work in the system is \( W^* \) defined by (4.5), and according to Lemma 4.2, as \( n \to \infty \), this work is increasingly concentrated to the right of the frontier \( \tilde{F}(n) \). One can use these observations to show that \( \hat{F}(n) \) has a limit, and this limit must be \( F^* = H^{-1}(W^*) \) so that

\[
\lim_{n \to \infty} \tilde{W}(n)(t)(F(n)(t), \infty) = H(F^*(t)) = W^*(t).
\]

In the limit, arrived work to the right of the frontier accounts for all work in the system. We summarize with the principal conclusions from [6].

**Theorem 4.4** As \( n \to \infty \),

\[
\hat{F}(n) \Rightarrow F^* \triangleq H^{-1}(W^*), \quad (4.14) \\
\hat{W}(n)(\cdot)(y, \infty) \Rightarrow H(y \vee F^*) \quad \forall y \in \mathbb{R}. \quad (4.15)
\]

The convergence in (4.15) is weak convergence of a sequence of measure-valued processes to a measure-valued process. In other words, the density of the limit of the measure-valued workload processes \( \hat{W}(n)(t) \) is \( (1 - G(x))\mathbb{I}_{[x \geq F^*(t)]} \), which is the density of the limit of \( V(n)(t) \) truncated at the random process \( F^*(t) \).

### 4.3 Reneging

We modify the real-time queueing system of the previous subsection by assuming that customers renege when they become late, i.e., a customer whose lead time reaches zero disappears from the queue never to return. This system has the same customer arrival process \( A(n) \), work arrival process \( V(n) \), and netput process \( N(n) \) as the system without reneging. However, in contrast to the EDF queueing system in Section 4.2, whose (scalar) workload process is Markov, the (scalar) workload process in the system with reneging is not Markov. This is because different customers reduce the workload by different amounts when they renege, and so it is necessary to keep track of the lead times and remaining service requirements of all customers. In the presence of reneging, we need the full measure-valued workload process in order to have a Markov system.
The workload process in the reneging system, denoted $W_{R}^{(n)}$, is less than or equal to the workload process $W^{(n)}$ of (4.2). For the reneging system, we scale the workload process to obtain (cf. (4.3))

$$
\hat{W}_{R}^{(n)}(t) = \frac{1}{\sqrt{n}} W_{R}^{(n)}(nt), \quad t \geq 0.
$$

For the reneging system, we define the measure-valued workload process (cf. (4.7))

$$(W_{R}^{(n)}(t)(B) = \left\{ \text{Work associated with customers in the reneging system at time } t \text{ with lead times in } B \right\})$$

and the scaled measure-valued workload process (cf. (4.10))

$$\hat{W}_{R}^{(n)}(t)(B) = \frac{1}{\sqrt{n}} W_{R}^{(n)}(nt)(\sqrt{n}B).$$

Here $B$ is an arbitrary Borel subset of $\mathbb{R}$. The measure-valued work arrival process $V^{(n)}$ and scaled measure-valued work arrival process $\hat{V}^{(n)}$ for the reneging system are the same as for the non-reneging system; these are given by (4.8) and (4.11). For the reneging system, the frontier is (cf. (4.9))

$$(F_{R}^{(n)}(t) = \begin{cases} \text{Largest lead time of any customer who has ever been in service in the reneging system, whether or not that customer is still present,} \\
\text{or } \sqrt{n} y^{*} - t \text{ if this quantity is larger than the former one} \end{cases})$$

Because the reneging system serves customers that have not yet been in service in the non-reneging system, we have $F^{(n)} \leq F_{R}^{(n)}$. The scaled frontier for the reneging system is (cf. (4.12))

$$\hat{F}_{R}^{(n)}(t) = \frac{1}{\sqrt{n}} F_{R}^{(n)}(t).$$

Recall the definition (1.6) of $\Lambda_{a}$ and the reflected Brownian motion $W^{*}$ of (4.5). The principal result of [11] is the following.

**Theorem 4.5** As $n \to \infty$,

$$\hat{W}_{R}^{(n)} \Rightarrow W_{R}^{*} \overset{\Delta}{=} \Lambda_{H(0)}(W^{*}), \quad \hat{F}_{R}^{(n)} \Rightarrow F_{R}^{*} \overset{\Delta}{=} H^{-1}(W_{R}^{*}), \quad \hat{W}_{R}^{(n)}(\cdot)(y, \infty) \Rightarrow H(y \vee F_{R}^{*}) \forall y \in \mathbb{R}. \quad (4.16)$$

We sketch the proof of Theorem 4.5. For this we introduce $\mathcal{M}$, the set of finite measures on the Borel subsets of $\mathbb{R}$. We endow $\mathcal{M}$ with the topology of weak convergence. We denote by $\mathcal{D}_{\mathcal{M}}[0, \infty)$ the set of functions from $[0, \infty)$ to $\mathcal{M}$ that are right-continuous and have left limits. We further define a mapping $\overline{\mathcal{A}}: \mathcal{D}_{\mathcal{M}}[0, \infty) \to \mathcal{D}_{\mathcal{M}}[0, \infty)$ by

$$\overline{\mathcal{A}}(\mu)(t)(-\infty, y] \overset{\Delta}{=} \left( \mu(t)(-\infty, y] - \sup_{s \in [0, t]} \mu(s)(-\infty, 0] \wedge \inf_{u \in [s, t]} \mu(u)(\mathbb{R}) \right)^{+}. \quad (4.19)$$
Consideration of (4.19) reveals that $\overline{\Lambda}(\mu)(t)$ is the measure on $\mathbb{R}$ that agrees with $\mu(t)$ except that it has all mass removed to the left of some point, no mass removed to the right of that point, and perhaps some of the mass removed at that point if there is a point mass there. The point in question is the supremum of those $y$ for which

$$
\mu(t)(-\infty, y] \leq \sup_{s \in [0, t]} \left[ \mu(s)(-\infty, 0] \land \inf_{u \in [s, t]} \mu(u)(\mathbb{R}) \right].
$$

The total amount of mass removed is almost the largest amount of “lateness” prior to time $t$, by which we mean $\sup_{s \in [0, t]} \mu(s)(-\infty, 0]$, but this is tempered by the fact that at some time between $t$ and the prior time $s$ when this maximal lateness was obtained, the system may have become empty. For example, if there is an $s_1 \in [0, t]$ and a $u_1 \in [s_1, t]$ such that

$$
sup_{s \in [0, t]} \left[ \mu(s)(-\infty, 0] \land \inf_{u \in [s, t]} \mu(u)(\mathbb{R}) \right] = \mu(s_1)(-\infty, 0] \land \inf_{u \in [s_1, t]} \mu(u)(\mathbb{R}) = \mu(u_1)(\mathbb{R}),
$$

then

$$
\overline{\Lambda}(\mu)(u_1)(\mathbb{R}) = \left( \mu(u_1)(\mathbb{R}) - sup_{s \in [0, u_1]} \left[ \mu(s)(-\infty, 0] \land \inf_{u \in [s, t]} \mu(u)(\mathbb{R}) \right] \right)^+ = 0;
$$

the system is empty at time $u_1$ and rather than subtracting mass $\mu(s_1)(-\infty, 0]$ from $\mu(t)$ to obtain $\overline{\Lambda}(\mu)(t)$, we subtract only $\mu(u_1)(\mathbb{R})$, the amount removed at time $u_1$ in order to create the empty system. We conclude the paper with the detailed Example 4.7 of the operation of $\overline{\Lambda}$ on a path of $\mathcal{W}^{(n)}$.

Unlike $\Lambda_n$ of (1.6), which maps real-valued functions to real-valued function, $\overline{\Lambda}$ maps measure-valued functions to measure-valued functions. To obtain a real-valued process, we define

$$
\mathcal{U}^{(n)}(t) \triangleq \overline{\Lambda}(\mathcal{W}^{(n)})(t),
$$

$$
\hat{U}^{(n)}(t) \triangleq \overline{U}^{(n)}(\mathbb{R})(t)
$$

$$
= \left( W^{(n)}(t) - sup_{s \in [0, t]} \left[ \mathcal{W}^{(n)}(s)(-\infty, 0] \land \inf_{u \in [s, t]} \mathcal{W}^{(n)}(u) \right] \right)^+.
$$

Scaling these relations, we obtain

$$
\hat{\tilde{U}}^{(n)}(t) \triangleq \frac{1}{\sqrt{n}} \hat{U}^{(n)}(nt)
$$

$$
= \left( \tilde{W}^{(n)}(t) - sup_{s \in [0, t]} \left[ \tilde{\mathcal{W}}^{(n)}(s)(-\infty, 0] \land \inf_{u \in [s, t]} \tilde{\mathcal{W}}^{(n)}(u) \right] \right)^+. \tag{4.20}
$$

Note that the processes $\tilde{W}^{(n)}(\cdot) = \tilde{\mathcal{W}}^{(n)}(\cdot)(\mathbb{R})$ and $\tilde{\mathcal{W}}^{(n)}$ appearing in (4.20) are for the non-reneging system. We take the limit as $n \to \infty$. Although $\overline{\Lambda}$ is not continuous on the set of all measure-valued processes, it is continuous on the set of processes that can result as the limit of $\mathcal{W}^{(n)}$. Therefore, we can use Theorems 4.1 and 4.4 and the continuous mapping theorem to obtain

$$
\hat{\tilde{U}}^{(n)}(t) \Rightarrow \left( W^*(t) - sup_{s \in [0, t]} \left[ (W^*(s) - H(0 \lor F^*(s))) \land \inf_{u \in [s, t]} W^*(u) \right] \right)^+, \tag{4.21}
$$

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where we have used the fact that
\[ \hat{W}_n(s)(-\infty, 0] = \hat{W}_n(s)(\mathbb{R}) - \hat{W}_n(s)(0, \infty) \Rightarrow W^*(s) - H(0 \lor F^*(s)). \]

Because \( H \) is nonincreasing,
\[ H(0 \lor F^*(s)) = H(0) \land H(F^*(s)) = H(0) \land W^*(s). \]

Therefore,
\[ W^*(s) - H(0 \lor F^*(s)) = W^*(s) - (H(0) \land W^*(s)) = (W^*(s) - H(0))^+. \]

Making this substitution in (4.21), we see that
\[ \hat{U}_n(T) \Rightarrow \Lambda H(0)(W^*). \]

In conclusion, we have defined
\[ \hat{U}_n(t) = \frac{1}{\sqrt{n}} \hat{W}_n(t)(\mathbb{R}) \quad \forall t \geq 0, \]

and taken the limit as \( n \to \infty \), and obtained (4.22).

The following lemma implies that the processes \( \hat{U}_n(t) \) and \( \hat{W}_n(t) \) have the same limit. In particular, (4.22) yields (4.16).

**Lemma 4.6** Let \( D^{(n)}(t) \) denote the work that arrives to the reneging system that has lead time upon arrival less than or equal to the frontier at the time of arrival and that ultimately reneges. Define \( \hat{D}^{(n)}(t) = \frac{1}{\sqrt{n}} D^{(n)}(nt) \). Then
\[ 0 \leq \hat{U}^{(n)} - \hat{W}_R^{(n)} \leq \hat{D}^{(n)} \Rightarrow 0, \]

where the limit in (4.24) is taken as \( n \to \infty \).

Just as with the non-reneging system, as \( n \to \infty \), work in the reneging system concentrates to the right of the frontier \( \hat{F}_R^{(n)} \). The remainder of the argument follows as in the derivation of (4.14) and (4.15). We know that the limiting scaled work in the system is \( W^* \), that this work must be concentrated to the right of the limiting frontier, and that work to the right of the frontier has never been in service and hence is just the work that has arrived. The limit of arrived work is given by Theorem 4.3, and (4.17) and (4.18) follow.

We do not attempt to prove Lemma 4.6 here. Instead, we illustrate it with the following example.

**Example 4.7** Consider a system realization in which
\[
\begin{align*}
&u_1^{(n)} = 1, v_1^{(n)} = 4, L_1^{(n)} = 3, S_1^{(n)} = 1, \\
u_2^{(n)} = 1, v_2^{(n)} = 4, L_2^{(n)} = 5, S_2^{(n)} = 2, \\
u_3^{(n)} = 3, v_3^{(n)} = 2, L_3^{(n)} = 1, S_3^{(n)} = 5, \\
u_4^{(n)} = 2, v_4^{(n)} = 1, L_4^{(n)} = 4, S_4^{(n)} = 7, \\
u_5^{(n)} = 2, v_5^{(n)} = 1, L_5^{(n)} = 1, S_5^{(n)} = 9.
\end{align*}
\]
Then using $\delta_s$ to denote a unit of mass at the point $s$, we have

$$W^{(n)}(t) = \begin{cases}
0, & 0 \leq t < 1, \\
(5-t)\delta_{4-t}, & 1 \leq t < 2, \\
(5-t)\delta_{4-t} + 4\delta_{7-t}, & 2 \leq t < 5, \\
(7-t)\delta_{0-t} + 4\delta_{7-t}, & 5 \leq t < 7, \\
(11-t)\delta_{7-t} + \delta_{11-t}, & 7 \leq t < 9, \\
2\delta_{-2} + \delta_1 + \delta_2, & t = 9.
\end{cases}$$

The measure $W^{(n)}(t)$ is shown for integer values of $t$ ranging between 1 and 9 in Figure 1.

We have $W^{(n)}(u) \geq 4$ for all $u \in [2, 8]$ and hence

$$K^{(n)}(t) \overset{\Delta}{=} \sup_{s \in [0,t]} \left[ W^{(n)}(s)(-\infty, 0] \wedge \inf_{u \in [s,t]} W^{(n)}(u) \right]$$

$$= \sup_{s \in [0,t]} W^{(n)}(s)(-\infty, 0]$$

$$= \begin{cases}
0, & 0 \leq t < 4, \\
1, & 4 \leq t < 7, \\
4, & 7 \leq t \leq 8.
\end{cases}$$

However, for $8 \leq t < 9$, we have $W^{(n)}(t) = 12 - t \leq 4$. For $t$ in this range, the supremum in the
The definition of $K^{(n)}(t)$ is attained at $s = 7$, and

$$K^{(n)}(t) = W^{(n)}(7)(-\infty, 0] \wedge \inf_{u \in [7, t]} W^{(n)} = 4 \wedge (12 - t) = 12 - t.$$  

For $t = 9$, we have $W^{(n)}(9) = 4 \neq 12 - t$. Nonetheless, the supremum in the definition of $K^{(n)}(9)$ is still attained at $s = 7$. Indeed,

$$K^{(n)}(9) = W^{(n)}(u)(-\infty, 0] \wedge \inf_{u \in [7, 9]} W^{(n)}(u) = 4 \wedge \left[ \inf_{u \in [7, 9]} (12 - t) \wedge 4 \right] = 3.$$  

In summary,

$$K^{(n)}(t) = \begin{cases} 0, & 0 \leq t < 4, \\ 1, & 4 \leq t < 7, \\ 4, & 7 \leq t \leq 8, \\ 12 - t, & 8 \leq t < 9. \end{cases}$$  

The measure $U^{(n)}(t)$ is obtained by subtracting mass $K^{(n)}(t)$ from the measure $W^{(n)}(t)$, working from left to right. This results in the formula

$$U^{(n)}(t) = \begin{cases} 0, & 0 \leq t < 1, \\ (5 - t)\delta_{4 - t}, & 1 \leq t < 2, \\ (5 - t)\delta_{4 - t} + 4\delta_{7 - t}, & 2 \leq t < 4, \\ (8 - t)\delta_{7 - t}, & 4 \leq t < 5, \\ (6 - t)\delta_{9 - t} + 4\delta_{7 - t}, & 5 \leq t < 6, \\ (10 - t)\delta_{7 - t}, & 6 \leq t < 7, \\ (8 - t)\delta_{11 - t}, & 7 \leq t < 8, \\ 0, & 8 \leq t < 9, \\ \delta_{2}, & t = 9. \end{cases}$$  

The measure $U^{(n)}(t)$ is shown for integer values of $t$ ranging between 1 and 9 in Figure 2.

The total mass in the $U^{(n)}$ system is

$$U^{(n)}(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 5 - t, & 1 \leq t < 2, \\ 9 - t, & 2 \leq t < 4, \\ 8 - t, & 4 \leq t < 5, \\ 10 - t, & 5 \leq t < 7, \\ 8 - t, & 7 \leq t < 8, \\ 0, & 8 \leq t < 9, \\ 1, & t = 9. \end{cases}$$  

This total mass path has jumps $\Delta U^{(n)}(1) = 4$, $\Delta U^{(n)}(2) = 4$, $\Delta U^{(n)}(4) = -1$, $\Delta U^{(n)}(5) = 2$, $\Delta U^{(n)}(7) = -2$ (the result of an arrival of mass 1 and the deletion of mass -3), and $U^{(n)}(9) = 1$.

We see that arriving mass to $U^{(n)}$ is not always placed at the lead time of the arriving customer. In particular, $U^{(n)}(5-) = 3\delta_2$, but $U^{(n)}(5) = \delta_1 + 4\delta_2$. The mass $v^{(n)}_3 = 2$ arriving at time 5 is distributed with one unit at $L^{(n)}_3 = 1$ and one unit at 2. Furthermore, the mass $v^{(n)}_5 = 1$ arriving at time $t = 9$, which begins a new busy period for $U^{(n)}$, is placed at 2 rather than at $L^{(n)}_5 = 1$.  

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Because of the failures of $U(n)$ to place all arriving masses at their lead times, the reneging system measure $W_R^{(n)}(t)$ is not $U^{(n)}(t)$ for $5 \leq t < 7$ and $t = 9$. The full formula for the reneging system is

$$W_R^{(n)}(t) = \begin{cases} 0, & 0 \leq t < 1, \\ (5-t)\delta_{1-t}, & 1 \leq t < 2, \\ (5-t)\delta_{1-t} + 4\delta_{7-t}, & 2 \leq t < 4, \\ (8-t)\delta_{7-t}, & 4 \leq t < 5, \\ (7-t)\delta_{5-t} + 3\delta_{7-t}, & 5 \leq t < 6, \\ (9-t)\delta_{7-t}, & 6 \leq t < 7, \\ (8-t)\delta_{11-t}, & 7 \leq t < 8, \\ 0, & 8 \leq t < 9, \\ \delta_1, & t = 9. \end{cases}$$

The measure $W_R^{(n)}(t)$ is shown for integer values of $t$ ranging between 1 and 9 in Figure 3.

Beginning at time $t = 4$, the reneging system begins serving the customer with lead time 3, and thus by time $t = 5$, this customer, whose lead time is now 2, requires only three remaining units of service. The customer arriving at time $t = 5$ with lead time 1 brings an additional two units of work. At time $t = 5$, the reneging system thus has five units of work, which agrees with $U^{(n)}(5) = 5$, but the mass in the reneging system is not distributed according to the measure $U^{(n)}(5)$. At time $t = 6$, an additional unit of work is deleted from the reneging system but not from the $U^{(n)}$ system, and so $W_R^{(n)}(6) = 3$, whereas $U^{(n)}(6) = 4$. This discrepancy is due to the deletion in the reneging system at time 6 of the customer who arrived at time $t = 5$, a customer.
who upon arrival was more urgent than the customer in service in the reneging system. The work associated with this customer upon arrival is counted in the process $D^{(n)}$ in Lemma 4.6.

Lemma 4.6 asserts that we always have $W^{(n)}_R(t) \leq U^{(n)}(t)$, and the inequality can be strict due to work that preempts the customer in service in the reneging system, but the difference between $W^{(n)}_R(t)$ and $U^{(n)}(t)$ is never more than the amount of such work deleted by the reneging system up to time $t$. □

Figure 3: Evolution of the reneging system $W^{(n)}_R$

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References


