Seventh International Workshop on Algebraic and Combinatorial Coding Theory
18-24 June, 2000, Bansko, Bulgaria *pp. 29–32*

Moments of spherical codes and designs

K. Andreev tanja@techno-link.com
Faculty of Mathematics and Informatics, Sofia University, 5 James Baucher str., Sofia, Bulgaria

S. Boumova silvi@moi.math.bas.bg
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 G. Bonchev str., 1113 Sofia, Bulgaria

P. Boyvalenkov peter@moi.math.bas.bg
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 8 G. Bonchev str., 1113 Sofia, Bulgaria

Abstract. We introduce and investigate certain invariants of spherical codes which may be useful in dealing with linear programming bounds for spherical codes and designs.

1 Definitions and main properties

An (n, M, s) *spherical code* is a nonempty finite subset of the n-dimensional Euclidean sphere $\mathbf{S}^{n-1}$ of cardinality $|C| = M$ and maximal inner product $s = s(C) = \max\{\langle x, y\rangle : x, y \in C, x \neq y\}$.

Let V_i, $i \geq 0$, be the space of homogeneous harmonic polynomials (considered as functions on $\mathbf{S}^{n-1}$) in n variables of total degree i. It is well known that

$$\dim(V_i) = r_i = \frac{n+2i-2}{n+i-2}\binom{n+i-2}{i}.$$

Let $\{v_{ij}(x) : 1 \leq j \leq r_i\}$ be an orthonormal basis of V_i. A *spherical τ-design* is a code $C \subset \mathbf{S}^{n-1}$ such that $\sum_{x\in C} v_{ij}(x) = 0$ for all $i = 1, 2, \ldots, \tau$ and all $j = 1, 2, \ldots, r_i$. The number τ is called *strength* of C. Finally, C is said to have an index i if $\sum_{x\in C} v_{ij}(x) = 0$ for all $j = 1, 2, \ldots, r_i$.

Another well known fact says that the formula

$$P_i^{(n)}(t) = \frac{1}{r_i}\sum_{j=1}^{r_i} v_i(x)v_j(y),$$

where $t = \langle x, y\rangle$, $i \geq 0$, gives nothing but the family of Gegenbauer polynomials (in particular, it depends on the inner product t of x and y but not on their particular choise). We omit the upper index in what follows.

For arbitrary real polynomial $f(t)$, let $f(t) = \sum_{i=0}^{k} f_i P_i(t)$ be its Gegenbauer expansion.

Definition 1 *For fixed spherical code* $C \subset \mathbf{S}^{n-1}$ *and any integer* $i \geq 1$*, the number*

$$M_i = \frac{1}{r_i}\sum_{j=1}^{r_i}(\sum_{x\in C} v_{ij}(x))^2$$

is called i-th moment of C*.*

The following theorem is obvious.

Theorem 1 *We have* $M_i \geq 0$ *for every* $i \geq 1$ *and* $M_1 = M_2 = \cdots = M_\tau = 0$ *if and only if* C *is a* τ*-design.* C *has index* i *if and only if* $M_i = 0$*.* C *is antipodal if and only if* $M_i = 0$ *for every odd* i*.*

One of the main initial sourses for obtaining bounds on parameters of codes and designs is the equality

$$f(1)|C| + \sum_{x,y\in C, x\neq y} f(\langle x,y\rangle) = f_0|C|^2 + \sum_{i=1}^{k} f_i M_i, \qquad (1)$$

which holds for any code $C \subset \mathbf{S}^{n-1}$ and any polynomial $f(t) = \sum_{i=0}^{k} f_i P_i(t)$.

Since obviously $M_i \geq 0$, the moments in (1) are usualy neglected. However, very often good codes have small strengths. This was our motivation to study the moments.

First of all, we observe that the moments do not depend on the choise of the bases $\{v_{ij}(x) : 1 \leq j \leq r_i\}$.

Theorem 2 $M_i = |C| + \sum_{x,y\in C, x\neq y} P_i(\langle x,y\rangle)$.

Proof. Set $f(t) = P_i(t)$ in (1).

2 Modified linear programming bounds

Using Theorem 2, one easily can calculate moments of known codes. In fact, it is enough to know the inner products and the distance distribution of the code under target. Therefore, one can calculate moments of many feasible classes of (good) codes and designs. However, we are also interested in moments of codes with uknown structure.

In this section we formulate four modifications of the linear programming bounds for spherical codes and designs (see [1, 4]). Their proofs are immediate from (1). Notice that all of them require preliminary information about moments of feasible codes (designs).

Theorem 3 *Let* $f(t)$ *be a real polynomial such that*
(A1) $f(t) \leq 0$ *for* $t \in [-1, s]$*;*

(A2) In the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i(t)$*, we have* $f_i \geq 0$ *for all* $i \in A = \{0, 1, \ldots, k\}$*;*
Let for any (n, M, s) *code we have* $M_k \geq \alpha_k > 0$ *for all* $k \in B \subset A$*. Then*

$$Mf(1) \geq M^2 f_0 + \sum_{k \in B} f_k \alpha_k.$$

Theorem 4 *Let* $f(t)$ *be a real polynomial such that*
(B1) $f(t) \geq 0$ *for* $t \in [-1, s]$*;*
(B2) In the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i(t)$*, we have* $f_i \geq 0$ *for all* $i \in A = \{0, 1, \ldots, k\}$*;*
Let for any (n, M, s) *code we have* $M_k \leq \beta_k$ *for all* $k \in B \subset A$*. Then*

$$Mf(1) \leq M^2 f_0 + \sum_{k \in B} f_k \beta_k.$$

Theorem 5 *Let* $f(t)$ *be a real polynomial such that*
(C1) $f(t) \leq 0$ *for* $t \in [-1, 1]$*;*
(C2) In the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i(t)$*, we have* $f_i \geq 0$ *for all* $i \in A = \{\tau + 1, \tau + 2, \ldots, k\}$*;*
Suppose also that for any τ*-design* $C \subset \mathbf{S}^{n-1}$ *of cardinality* M *we have* $M_k \geq \alpha_k > 0$ *for all* $k \in B \subset A$*. Then*

$$Mf(1) \leq M^2 f_0 + \sum_{k \in B} f_k \alpha_k.$$

Theorem 6 *Let* $f(t)$ *be a real polynomial such that*
(D1) $f(t) \geq 0$ *for* $t \in [-1, 1]$*;*
(D2) In the Gegenbauer expansion $f(t) = \sum_{i=0}^{k} f_i P_i(t)$*, we have* $f_i \geq 0$ *for all* $i \in A = \{\tau + 1, \tau + 2, \ldots, k\}$*;*
Suppose also that for any τ*-design* $C \subset \mathbf{S}^{n-1}$ *of cardinality* M *we have* $M_k \leq \beta_k$ *for all* $k \in B \subset A$*. Then*

$$Mf(1) \geq M^2 f_0 + \sum_{k \in B} f_k \beta_k.$$

It is clear by Theorem 2 that the upper bounds β_k exist for every k. Better bounds can be obtained by using suitable polynomials in (1). Actually, any polynomial which does not change its sign in $[-1, s]$ (or $[-1, 1]$ for designs) gives by (1) a linear inequality for the relevant moments. A set of such inequalities can be used then in usual linear programming (i.e. it can be investigated by the simplex method). We used the software package SCOD [3, 2] in order to obtain some bounds on the moments of putative $(4, 25, 0.5)$ code.

3 Some examples

As usually, the design problem allows more detailed investigation. This is because the conditions (C1) and (D1) in Theorems 5 and 6 above are in fact stronger than necessary. Indeed, for designs of small cardinalities one usually knows that all inner products belong to some intervals $[a, b] \subset [-1, 1]$. We illustrate how this helps for obtaining better bounds on moments of spherical designs.

Example 1. *Let $C \subset \mathbf{S}^{n-1}$ be a spherical 4-design. Then $M_i = 0$ for $1 \leq i \leq 4$ and the first "interesting" moment is M_5. Consider the polynomials*

$$f(t) = (t - \alpha)(t^2 + at + b)^2,$$

where a and b are parameters to be optimized and α is a lower (resp. upper) bound on the inner products $\langle x, y\rangle$, $x, y \in C$, $x \neq y$. Then by (1) we obtain

$$f_5 M_5 \geq |C|(f(1) - f_0|C|) = |C|F(\alpha, a, b)$$

or

$$f_5 M_5 \leq |C|(f(1) - f_0|C|) = |C|F(\alpha, a, b),$$

respectively. Here $f_5 = (n^2 - 1)/(n + 2)(n + 4)$ does not depend on C, α, a and b, and

$$F(\alpha, a, b) = (1 - \alpha)(1 + a + b)^2 - |C|\Big(- \alpha b^2 + \frac{2ab - \alpha(a^2 + 2b)}{n} + \frac{3(2a - \alpha)}{n(n + 2)}\Big).$$

For particular values of α, we have to optimize with respect to a and b. The first open case is $n = 3$, $|C| = 10$ (i.e., it is ever unknown if there exists a 10-point 4-design in three dimesnions). Since all inner products of such a design must belong to $[-\sqrt{23/27}, 0.466)$, we obtain that $22.1 \leq M_5 \leq 33.6$.

References

[1] P. Delsarte, J.-M. Goethals, J.J. Seidel, Spherical codes and designs, *Geom. Dedicata* 6, 1977, 363-388.

[2] P.S. Kazakov, Applications of polynomials for CRC and spherical codes, PhD Dissertation, TU Delft, the Netherlands, 2000.

[3] P. Boyvalenkov, S. Bumova, D. Danev, P. Kazakov, A program for obtaining LPB for spherical codes, in *Proc. Fifth Intern. Workshop on Algebraic and Combinatorial Coding Theory*, Sozopol, Bulgaria 1996, 68-70.

[4] V.I. Levenshtein, Universal bounds for codes and designs, Chapter 6 (499-648) in *Handbook of Coding Theory*, Eds. V. Pless and W.C. Huffman, Elsevier Science B.V., 1998.