Exponentially affine martingales, affine measure changes and exponential moments of affine processes

Jan Kallsen*   Johannes Muhle-Karbe†

Abstract

We consider local martingales of exponential form $M = e^X$ or $\mathscr{E}(X)$ where X denotes one component of a multivariate affine process. We give a weak sufficient criterion for M to be a true martingale. As a first application, we derive a simple sufficient condition for absolute continuity of the laws of two given affine processes. As a second application, we study whether the exponential moments of an affine process solve a generalized Riccati equation.

Key words: Affine processes, exponential martingale, uniform integrability, change of measure, exponential moments, generalized Riccati equation

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1 Introduction

Affine processes play an important role in stochastic calculus and its applications e.g. in mathematical finance (cf. [3, 4, 6, 7, 14]). Their popularity for modelling purposes is probably due to their combination of flexibility and mathematical tractability. This paper studies the following questions concerning exponentials of affine processes.

1. Suppose that the exponential of an affine process is a local martingale. Under what conditions is it a true martingale?

2. Suppose that two parameter sets of affine processes are given. Do they correspond to the same process under equivalent probability measures?

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*Mathematisches Seminar, Christian-Albrechts-Universität zu Kiel, Christian-Albrechts-Platz 4, 24098 Kiel, Germany, (e-mail: kallsen@math.uni-kiel.de).

†Fakultät für Mathematik, Universität Wien, Nordbergstraße 15, 1090 Wien, Austria, (e-mail: johannes.muhle-karbe@univie.ac.at).

3. Under what condition is the p-th exponential moment of an affine process given as the solution to a generalized Riccati equation?

The first question is of interest in statistics and mathematical finance, where such exponentials denote density and price processes. General criteria as the Novikov condition or its generalizations to processes with jumps in [15] and [17] are generally far from necessary. Less restrictive criteria have been obtained by making subtle use of e.g. the Markovian structure of the process. In [11] and [4] it is shown that in the context of bivariate affine diffusions, any exponential local martingale is a true martingale. Similarly, [18, 5] contain conditions for the exponential of a diffusion with and without jumps to be a martingale. Below in Section 3 we present weak sufficient conditions which are taylor-made for affine processes and easy to verify.

The second question is motivated from statistics and finance as well. Applied to finance, one law plays the role of the physical probability measure whereas the other is used as a risk-neutral measure for derivative pricing. In order to be consistent with arbitrage theory, these laws must be equivalent. In Section 4 we derive sufficient conditions which are based on the results of Section 3. On the one hand, these extend the results of [12] on Lévy processes. On the other hand, they resemble results of [5] applied to the affine case, however with sometimes less restrictive moment conditions.

As a function of t, the characteristic function $E(\exp(iu^\top X_t))$, $u \in \mathbb{R}^d$, of an $\mathbb{R}^d$-valued affine process X solves a generalized Riccati equation as it is shown in great generality in [6] and [9]. Morally speaking, the same should hold for real exponential moments $E(\exp(p^\top X_t))$, $p \in \mathbb{R}^d$. Statements in [6] suggest that this may hold for arbitrary affine processes but the paper does not seem to provide an applicable condition. We study this question in Section 5.

We generally use the notation of [12]. By $X \bullet Y$ we denote the stochastic integral of X with respect to Y. For any semimartingale X we write $\mathscr{E}(X)$ for the *stochastic exponential* of X (cf. [12, I.4.61]). Moreover, $\mathscr{L}(Y)$ denotes the *stochastic logarithm* of a semimartingale Y with $Y, Y_- \neq 0$, see [12, II.8.3]. The identity process is written as I, i.e. $I_t = t$. When dealing with stochastic processes and Lévy-Khintchine triplets, superscripts generally refer to coordinates of a vector rather than powers. The set $\mathbb{N}$ includes 0.

The following section summarizes facts on semimartingales and affine processes that are needed in the sequel. For more details see e.g. [6, 9, 12, 14]. The appendix contains some supplementary results in this context.

2 Semimartingale calculus and affine processes

Often affine processes are introduced as Markov processes whose characteristic function is of exponentially affine form. We study them from the point of view of semimartingale theory. In this context they correspond to processes with affine characteristics.

2.1 Semimartingale calculus

We call the derivative of semimartingale characteristics in the sense of [12] differential characteristics:

Definition 2.1 Let X be an $\mathbb{R}^d$-valued semimartingale with characteristics (B, C, ν) relative to some truncation function $h : \mathbb{R}^d \to \mathbb{R}^d$. If there exist some predictable $\mathbb{R}^d$-valued process b, some predictable $\mathbb{R}^{d \times d}$-valued process c whose values are nonnegative, symmetric matrices, and some transition kernel F from $(\Omega \times \mathbb{R}_+, \mathscr{P})$ into $(\mathbb{R}^d, \mathscr{B}^d)$ where $\mathscr{P}$ denotes the σ-algebra of predictable sets, such that

$$B_t = b \bullet I_t, \quad C_t = c \bullet I_t, \quad \nu([0, t] \times G) = F(G) \bullet I_t \text{ for } G \in \mathscr{B}^d,$$

we call (b, c, F) *differential characteristics* of X relative to h and we denote them by ∂X.

Recall that $b \bullet I_t$ means $\int_0^t b_s ds$ etc. because $I_t = t$. Differential characteristics of Markov processes are deterministic functions of the current state of the process. This leads to the notion of a martingale problem in the following sense.

Definition 2.2 Suppose that P_0 is a distribution on $\mathbb{R}^d$ and mappings $\beta : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^d$, $\gamma : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}^{d \times d}$, $\varphi : \mathbb{R}^d \times \mathbb{R}_+ \times \mathscr{B}^d \to \mathbb{R}_+$ are given. We call $(\Omega, \mathscr{F}, \mathbf{F}, P, X)$ *solution* to the *martingale problem* related to P_0 and (β, γ, φ) if X is a semimartingale on $(\Omega, \mathscr{F}, \mathbf{F}, P)$ such that $P^{X_0} = P_0$ and $\partial X = (b, c, F)$ with

$$
\begin{align}
b_t(\omega) &= \beta(X_{t-}(\omega), t), \tag{2.1} \\
c_t(\omega) &= \gamma(X_{t-}(\omega), t), \tag{2.2} \\
F_t(\omega, G) &= \varphi(X_{t-}(\omega), t, G). \tag{2.3}
\end{align}
$$

One may also call the distribution P^X of X *solution* to the martingale problem. Since we consider only càdlàg solutions, P^X is a probability measure on the Skorohod or canonical path space $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ of $\mathbb{R}^d$-valued càdlàg functions on $\mathbb{R}_+$ endowed with its natural filtration (cf. [12, Chapter VI]). When dealing with this space, we denote by X the canonical process, i.e. $X_t(\alpha) = \alpha(t)$ for $\alpha \in \mathbb{D}^d$. In any case, *uniqueness* of the solution refers only to the law P^X because processes on different probability spaces cannot reasonably be compared otherwise.

For later use we consider the effect of stopping on the characteristics and differential characteristics:

Lemma 2.3 *Let τ be a stopping time and X an $\mathbb{R}^d$-valued semimartingale with characteristics (B, C, ν). Then the stopped process X^τ has characteristics $(B^\tau, C^\tau, \nu^\tau)$, where ν^τ here refers to the random measure given by*

$$1_G * \nu^\tau := 1_{G \cap [\![0, \tau]\!]} * \nu, \quad \forall G \in \mathscr{P}.$$

If X admits differential characteristics (b, c, F), then X^τ has differential characteristics $\partial X^\tau = (b 1_{[\![0, \tau]\!]}, c 1_{[\![0, \tau]\!]}, F(dx) 1_{[\![0, \tau]\!]})$.

PROOF. By [12, II.2.42] we have $A(u) \in \mathscr{M}_{\text{loc}}$ for $u \in \mathbb{R}^d$, where

$$A(u) := e^{iu^\top X} - e^{iu^\top X_-} \bullet \left(iu^\top B - \frac{1}{2} u^\top C u + \int_{[0,\cdot] \times \mathbb{R}^d} (e^{iu^\top x} - 1 - iu^\top h(x)) \nu(d(t,x)) \right).$$

Since $\mathscr{M}_{\text{loc}}$ is stable under stopping, we have $A^\tau \in \mathscr{M}_{\text{loc}}$. Moreover, [12, I.4.37] yields

$$\begin{aligned} A^\tau(u) &= e^{iu^\top X^\tau} \\ &\quad - e^{iu^\top X_-^\tau} \bullet \left(iu^\top B^\tau - \frac{1}{2} u^\top C^\tau u + \int_{[0,\cdot] \times \mathbb{R}^d} (e^{iu^\top x} - 1 - iu^\top h(x)) \nu^\tau(d(t,x)) \right). \end{aligned}$$

Again by [12, II.2.42] the characteristics of X^τ have the desired form. The second claim now follows from $(b1_{[0,\tau]}) \bullet I = B^\tau$, $(c1_{[0,\tau]}) \bullet I = C^\tau$ and

$$(F(G)1_{[0,\tau]}) \bullet I_t = \nu^\tau([0,t] \times G)$$

for all $G \in \mathscr{B}^d$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

2.2 Time-inhomogeneous affine processes

From now on, we only consider *affine martingale problems*, where the differential characteristics are affine functions of X_{t-} in the following sense:

$$\beta((x_1, ..., x_d), t) = \beta_0(t) + \sum_{j=1}^{d} x_j \beta_j(t), \tag{2.4}$$

$$\gamma((x_1, ..., x_d), t) = \gamma_0(t) + \sum_{j=1}^{d} x_j \gamma_j(t), \tag{2.5}$$

$$\varphi((x_1, ..., x_d), t, G) = \varphi_0(t, G) + \sum_{j=1}^{d} x_j \varphi_j(t, G), \tag{2.6}$$

where $(\beta_j(t), \gamma_j(t), \varphi_j(t))$, $j = 0, ..., d$, $t \in \mathbb{R}_+$ are given Lévy-Khintchine triplets on $\mathbb{R}^d$. If the triplets do not depend on t, we are in the setting of [6], where results on affine Markov processes yield conditions for the existence of a unique solution to this problem (cf. [14]). In the time-inhomogeneous case we turn to the corresponding results of [9], namely Theorems 2.13 and 2.14.

However, we require the solution process to be a semimartingale in the usual sense, i.e. with finite values for all $t \in \mathbb{R}_+$. In [9] it is established that this is the case if the Markov process in question is *conservative*, but it does not contain analogues to the criteria for the homogeneous case in [6]. Therefore we extend [6, Lemma 9.2] to the time-inhomogeneous case, which is done in the appendix.

Unlike most results in semimartingale theory, the conditions in [9] depend on the choice of the truncation function on $\mathbb{R}^d$. From now on, we assume it to be of the form $h =$

$(h_1, ..., h_d)$ with

$$h_k(x) := \chi(x_k) := \begin{cases} 0 & \text{if } x_k = 0, \\ (1 \wedge |x_k|) \frac{x_k}{|x_k|} & \text{otherwise.} \end{cases}$$

Definition 2.4 Let $d \in \mathbb{N}\backslash\{0\}$. Lévy-Khintchine triplets $(\beta_j(t), \gamma_j(t), \varphi_j(t))$, $j = 0, ..., d$, $t \in \mathbb{R}_+$, are called *strongly admissible* if there exists $m \in \mathbb{N}$, $m \leq d$ such that, for $t \in \mathbb{R}_+$,

$$\beta_j^k(t) - \int h_k(x)\varphi_j(t, dx) \geq 0 \quad \text{if} \quad 0 \leq j \leq m, \quad 1 \leq k \leq m, \quad k \neq j;$$
$$\varphi_j\big(t, (\mathbb{R}_+^m \times \mathbb{R}^{d-m})^c\big) = 0 \quad \text{if} \quad 0 \leq j \leq m;$$
$$\int h_k(x)\varphi_j(t, dx) < \infty \quad \text{if} \quad 0 \leq j \leq m, \quad 1 \leq k \leq m, \quad k \neq j;$$
$$\gamma_j^{kl}(t) = 0 \quad \text{if} \quad 0 \leq j \leq m, \quad 1 \leq k, l \leq m \quad \text{unless } k = l = j;$$
$$\beta_j^k(t) = 0 \quad \text{if} \quad j \geq m+1, \quad 1 \leq k \leq m;$$
$$\gamma_j(t) = 0 \quad \text{if} \quad j \geq m+1;$$
$$\varphi_j(t, \cdot) = 0 \quad \text{if} \quad j \geq m+1$$

and if the following continuity conditions are satisfied:

- $\beta_j(t), \gamma_j(t)$ are continuous in $t \in \mathbb{R}_+$ for $0 \leq j \leq d$,

- $h_k(x)\varphi_j(t, dx)$ is weakly continuous on $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\backslash\{0\}$ for $0 \leq j \leq d, 1 \leq k \leq m$ with $k \neq j$,

- $h_k(x)^2\varphi_j(t, dx)$ is weakly continuous on $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\backslash\{0\}$ for $0 \leq j \leq d$ and $k \geq m+1$ or $k = j$,

i.e. for $s \to t \in \mathbb{R}_+$ and any bounded continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we have

$$\int f(x)h_k(x)\varphi_j(s, dx) \to \int f(x)h_k(x)\varphi_j(t, dx) \text{ if } 0 \leq j \leq d, 1 \leq k \leq m, k \neq j,$$
$$\int f(x)h_k(x)^2\varphi_j(s, dx) \to \int f(x)h_k(x)^2\varphi_j(t, dx) \text{ if } 0 \leq j \leq d, k \geq m+1 \text{ or } k = j.$$

Remark 2.5 If the Lévy-Khintchine triplets do not depend on t, this definition is consistent with [14, Definition 4]. In this case, the attribute *strongly* can and will be dropped because it refers to continuity in t. In particular, the choice of the truncation function does not matter. In the time-inhomogeneous case however, the continuity conditions depend on the choice of the truncation function. Nevertheless, the function h defined explicitly above can be replaced by an arbitrary continuous truncation function $\widetilde{h}$ satisfying $|\widetilde{h}| \geq \varepsilon > 0$ outside of some neighbourhood of 0.

In view of Lemma A.1 below, [9, Theorems 2.13 and 2.14] can immediately be re-phrased as an existence and uniqueness result for affine martingale problems, which extends [14, Theorem 3.1] to the time-inhomogeneous case.

Theorem 2.6 (Affine semimartingales) *Let $(\beta_j(t), \gamma_j(t), \varphi_j(t))$, $j = 0, ..., d$, $t \in \mathbb{R}_+$ be strongly admissible Lévy-Khintchine triplets and denote by ψ_j the corresponding Lévy exponents*

$$\psi_j(t, u) = u^\top \beta_j(t) + \frac{1}{2} u^\top \gamma_j(t) u + \int (e^{u^\top x} - 1 - u^\top h(x)) \varphi_j(t, dx).$$

Suppose in addition that

$$\sup_{t \in [0,T]} \int_{\{x_k > 1\}} x_k \varphi_j(t, dx) < \infty \quad \text{for } j, k = 1, \ldots, m, \quad \forall T \in \mathbb{R}_+. \tag{2.7}$$

Then the affine martingale problem related to (β, γ, φ) and some initial distribution P_0 on $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$ has a solution P on $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ such that X is $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$-valued. For $0 \le t \le T$ the corresponding conditional characteristic function is given by

$$E\left(e^{i\lambda^\top X_T} \middle| \mathscr{D}_t\right) = \exp\left(\Psi^0(t, T, i\lambda) + \Psi^{(1,...,d)}(t, T, i\lambda)^\top X_t\right), \quad \forall \lambda \in \mathbb{R}^d, \tag{2.8}$$

where

$$\Psi^0(t, T, u) = \int_t^T \psi_0(s, \Psi^{(1,...,d)}(s, T, u)) ds \tag{2.9}$$

and $\Psi^{(1,...,d)} := (\Psi^1, \ldots, \Psi^d)$ solves the following generalized Riccati equations:

$$\Psi^{(1,...,d)}(T, T, u) = u, \quad \frac{d}{dt} \Psi^j(t, T, u) = -\psi_j(t, \Psi^{(1,...,d)}(t, T, u)), \quad j = 1, ..., d. \tag{2.10}$$

Moreover, if $(\Omega', \mathscr{F}', \mathbf{F}', P', X')$ is another solution to the affine martingale problem, the distributions of X and X' coincide, i.e. $P^{X'} = P$.

PROOF. This follows from [9, Theorems 2.13, 2.14] and Lemma A.1 below along the lines of the proof of [14, Theorem 3.1]. □

As is well known, the stochastic exponential of a real-valued Lévy process X with $\Delta X > -1$ is the ordinary exponential of another Lévy process and vice versa. A similar statement holds for components of affine processes:

Lemma 2.7 *Let X be an $\mathbb{R}^d$-valued semimartingale with affine differential characteristics relative to strongly admissible Lévy-Khintchine triplets $(\beta_j(t), \gamma_j(t), \varphi_j(t))$, $0 \le j \le d$, $t \in \mathbb{R}_+$. Let $i \in \{1, \ldots, d\}$. Then the differential characteristics of*

$$(X, \widetilde{X}^i) := (X, \mathscr{L}(\exp(X^i)))$$

are affine with $\widetilde{m} = m$, $\widetilde{d} = d + 1$, relative to strongly admissible Lévy-Khintchine triplets $(\widetilde{\beta}_j(t), \widetilde{\gamma}_j(t), \widetilde{\varphi}_j(t))$, $0 \le j \le d + 1$, $t \in \mathbb{R}_+$, where $(\widetilde{\beta}_{d+1}(t), \widetilde{\gamma}_{d+1}(t), \widetilde{\varphi}_{d+1}(t)) = (0, 0, 0)$

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and

$$\widetilde{\beta}_j(t) = \begin{pmatrix} \beta(t) \\ \beta_j^i(t) + \frac{1}{2}\gamma_j^{ii}(t) + \int(\chi(e^{x_i}-1) - \chi(x_i))\varphi_j(t,dx) \end{pmatrix},$$

$$\widetilde{\gamma}_j^{kl}(t) = \begin{cases} \gamma_j^{kl}(t) & \textit{for } k,l = 1,\ldots,d, \\ \gamma_j^{il}(t) & \textit{for } k = d+1, \quad l = 1,\ldots,d, \\ \gamma_j^{ki}(t) & \textit{for } k = 1,\ldots,d, \quad l = d+1, \\ \gamma_j^{ii}(t) & \textit{for } k,l = d+1, \end{cases}$$

$$\widetilde{\varphi}_j(t,G) = \int 1_G(x, e^{x_i}-1)\varphi_j(t,dx), \quad \forall G \in \mathscr{B}^{d+1},$$

for $0 \le j \le d$. Furthermore we have $\exp(X^i) = \exp(X_0^i)\mathscr{E}(\widetilde{X}^i)$.

PROOF. The characteristics can be computed with [14, Propositions 2 and 3]. Strong admissibility of the triplets $(\widetilde{\beta}_j, \widetilde{\gamma}_j, \widetilde{\varphi}_j)$ follows immediately from strong admissibility of $(\beta_j, \gamma_j, \varphi_j)$ because the mapping $x \mapsto \frac{\chi(e^{x_i}-1)-\chi(x_i)}{\chi(x_i)^2}$ is bounded and continuous on $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\setminus\{0\}$. $\square$

3 Exponentially affine martingales

In this section we provide criteria for the exponential of a component of an affine process to be a martingale. We start with a general sufficient condition which is proved in Section 3.2. In Sections 3.3 and 3.4 we apply this general result to the time-homogeneous case and to processes with independent increments, respectively.

3.1 Time-inhomogeneous exponentially affine martingales

Let X be an $\mathbb{R}^d$-valued semimartingale with affine differential characteristics relative to strongly admissible Lévy-Khintchine triplets $(\beta_j(t), \gamma_j(t), \varphi_j(t))$, $0 \le j \le d$, $t \in \mathbb{R}_+$. The following result is proved in Section 3.2.

Theorem 3.1 *Suppose that for some $1 \le i \le d$ and $T \in \mathbb{R}_+$ the following holds:*

1. *$\varphi_j(t, \{x \in \mathbb{R}^d : x_i < -1\}) = 0$ for $j = 0,\ldots,m$, $\forall t \in [0,T]$,*

2. *$\int_{\{x_i > 1\}} x_i \varphi_j(t,dx) < \infty$ for $j = 0,\ldots,m$, $\forall t \in [0,T]$,*

3. *$\beta_j^i(t) + \int(x_i - h_i(x))\varphi_j(t,dx) = 0$ for $j = 0,\ldots,d$, $\forall t \in [0,T]$,*

4. *the measure $h_k(x)x_i\varphi_j(t,dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\setminus\{0\}$ is weakly continuous in $t \in [0,T]$ for $j = 1,\ldots,m$ and $k = 1,\ldots,d$.*

5. *$\sup_{t \in [0,T]} \int_{\{x_k > 1\}} x_k(1 + x_i)\varphi_j(t,dx) < \infty$ for $j,k = 1,\ldots,m$.*

Then the stopped process $\mathscr{E}(X^i)^T$ is a martingale.

Condition 1 ensures that $\mathscr{E}(X^i)$ does not jump to negative values. Condition 2 is needed for the integral in Condition 3 to be finite. Condition 3 in turn means that $(X^i)^T$ and hence also $\mathscr{E}(X^i)^T$ have zero drift, i.e. they are σ-martingales (cf. [13, Lemmas 3.1 resp. 3.3]). The continuity condition 4 is needed to apply the results of [9]. It holds automatically in the time-homogeneous case (cf. Corollary 3.9). The crucial nontrivial assumption is the last one. The origin of this moment condition is discussed in Section 3.2.

From Theorem 3.1 we can obtain a similar result on the entire real line:

Corollary 3.2 *Suppose that for some $1 \leq i \leq d$ and all $t \in \mathbb{R}_+$ the following holds:*

1. *$\varphi_j(t, \{x \in \mathbb{R}^d : x_i < -1\}) = 0$ for $j = 0, \ldots, m$, $\forall t \in \mathbb{R}_+$,*

2. *$\int_{\{x_i > 1\}} x_i \varphi_j(t, dx) < \infty$ for $j = 0, \ldots, m$, $\forall t \in \mathbb{R}_+$,*

3. *$\beta_j^i(t) + \int (x_i - h_i(x))\varphi_j(t, dx) = 0$ for $j = 0, \ldots, d$, $\forall t \in \mathbb{R}_+$,*

4. *the measure $h_k(x)x_i\varphi_j(t, dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\setminus\{0\}$ is weakly continuous in t for $j = 1, \ldots, m$ and $k = 1, \ldots, d$.*

5. *$\sup_{t \in [0,T]} \int_{\{x_k > 1\}} x_k(1 + x_i)\varphi_j(t, dx) < \infty$ for $j, k = 1, \ldots, m$, $\forall T \in \mathbb{R}_+$.*

Then $\mathscr{E}(X^i)$ is a martingale.

PROOF. By Theorem 3.1, $\mathscr{E}(X^i)^T$ is a martingale for all $T \in \mathbb{R}_+$, which implies that $\mathscr{E}(X^i)$ is a martingale as well. $\square$

Example 3.3 If X is continuous, Conditions 1–5 above reduce to $\beta_j^i = 0$, $j = 0, \ldots, d$, i.e. essentially to assuming that $\mathscr{E}(X^i)$ is a local martingale. This applies e.g. to the asset price in the stochastic volatility model introduced by Heston [10] (cf. [14] for the differential characteristics).

We also obtain an analogue of Theorem 3.1 for ordinary exponentials:

Corollary 3.4 *Suppose that for some $1 \leq i \leq d$ and $T \in \mathbb{R}_+$ the following holds:*

1. *$E(e^{X_0^i}) < \infty$,*

2. *$\int_{\{x_i > 1\}} e^{x_i}\varphi_j(t, dx) < \infty$, $j = 0, \ldots, m$, $\forall t \in [0, T]$,*

3. *$\beta_j^i(t) + \frac{1}{2}\gamma_j^{ii}(t) + \int (e^{x_i} - 1 - h_i(x))\varphi_j(t, dx) = 0$, $j = 0, \ldots, d$, $\forall t \in [0, T]$,*

4. *the measure $h_k(x)(e^{x_i} - 1)\varphi_j(t, dx)$ on $(\mathbb{R}_+^m \times \mathbb{R}^{d-m})\setminus\{0\}$ is weakly continuous in $t \in [0, T]$ for $j = 1, \ldots, m$ and $k = 1, \ldots, d$,*

5. *$\sup_{t \in [0,T]} \int_{\{x_k > 1\}} x_k e^{x_i}\varphi_j(t, dx) < \infty$ for $j, k = 1, \ldots, m$.*

Then the stopped process $(e^{X^i})^T$ is a martingale.

PROOF. By [14, Proposition 3] and [13, Lemma 3.1] the process $\exp(X^i)^T$ is a σ-martingale. From [13, Proposition 3.1] it follows that it is a supermartingale, in particular it is integrable. We have $\exp(X^i) = e^{X_0^i}\mathscr{E}(\widetilde{X}^i)$ for $\widetilde{X}^i$ as in Lemma 2.7. $\mathscr{E}(\widetilde{X}^i)^T$ is a martingale by Theorem 3.1. Since $e^{X_0^i}$ is integrable, we have

$$E\big(e^{X_t^i}\big) = E\big(e^{X_0^i}E\big(\mathscr{E}(\widetilde{X}^i)_t|\mathscr{F}_0\big)\big) = E\big(e^{X_0^i}\big) < \infty.$$

This yields that e^{X_i} is a martingale as well. $\qquad\square$

Of course an analogue of Corollary 3.2 holds for ordinary exponentials as well.

3.2 Proof of Theorem 3.1

Set $M := \mathscr{E}(X^i)^T$. In view of [13, Lemma 3.1], Conditions 2 and 3 imply that X^i is a σ-martingale. By [13, Lemma 3.3] this shows that M is a σ-martingale, too. Condition 1 implies $\Delta X^i \geq -1$ on $[0, T]$, which in turn yields $M \geq 0$. Since any nonnegative σ-martingale is a supermartingale (cf. [13, Proposition 3.1]), it remains to show that $E(M_T) = 1$. Since this property only depends on the law of X, we can assume without loss of generality that X is the canonical process on the canonical path space.

If M is a martingale, we can use it as the density process of a locally absolutely continuous measure change and employ Girsanov's theorem to calculate the characteristics of the canonical process under this new measure. In this proof the fundamental idea is to work in the opposite direction: we *define* the triplets as motivated by Girsanov and prove that there is a probability measure Q that endows the canonical process with these characteristics. There we need the crucial moment condition 5. Next, we establish that this new measure is locally absolutely continuous with respect to the original probability measure, by using a certain uniqueness property of the martingale problems in question. Hence a density process exists. The final step of the proof is to show that this density process coincides with M. Related approaches are taken e.g. in [4, 5, 11, 18].

Lemma 3.5 *For $j = 0, \ldots, d$ and $t \in \mathbb{R}_+$ set*

$$\beta_j^*(t) = \beta_j(t \wedge T) + \gamma_j^{\cdot i}(t \wedge T) + \int x_i h(x)\varphi_j(t \wedge T, dx), \qquad (3.1)$$

$$\gamma_j^*(t) = \gamma_j(t \wedge T), \qquad (3.2)$$

$$\varphi_j^*(t, G) = \int 1_G(x)(1 + x_i)\varphi_j(t \wedge T, dx), \quad \forall G \in \mathscr{B}^d. \qquad (3.3)$$

Under Conditions 1–4 of Theorem 3.1 this defines strongly admissible Lévy-Khintchine triplets. If Condition 5 holds as well, then there is a unique solution Q to the corresponding affine martingale problem on $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ with any fixed initial distribution Q_0 on $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$.

PROOF. In view of Condition 5 and Theorem 2.6, it suffices to show that $(\beta_j^*(t), \gamma_j^*(t), \varphi_j^*(t))$ are strongly admissible Lévy-Khintchine triplets. Let $0 \leq t \leq T$. By Condition 2 the

integral in (3.1) exists. The equivalence of $\varphi_j^*(t, dx)$ and $\varphi_j(t, dx)$ implies $\varphi_j^*(\{0\}) = 0$ and we have

$$\int (1 \wedge |x|^2) \varphi_j^*(t, dx) = \int (1 \wedge |x|^2) \varphi_j(t \wedge T, dx) + \int (1 \wedge |x|^2) x_i \varphi_j(t \wedge T, dx) < \infty$$

because $\varphi_j(t)$ is a Lévy measure and by Condition 2. Therefore $(\beta_j^*(t), \gamma_j^*(t), \varphi_j^*(t))$ are Lévy-Khintchine triplets. Now let $0 \leq j \leq m, 1 \leq k \leq m, k \neq j$. Then

$$\beta_j^{*k}(t) - \int h_k(x) \varphi_j^*(t, dx) = \beta_j^k(t \wedge T) - \int h_k(x) \varphi_j(t, dx) \geq 0$$

because of the first and fourth admissibility condition for the original triplets $(\beta_j, \gamma_j, \varphi_j)$. From the second admissibility condition and by equivalence of $\varphi_j(t, dx)$ and $\varphi_j^*(t, dx)$ we obtain $\varphi_j^*(t, (\mathbb{R}_+^m \times \mathbb{R}^{d-m})^c) = 0$. Moreover, Condition 2 and the third condition on the original triplets yield

$$\int h_k(x) \varphi_j^*(t, dx) = \int h_k(x)(1 + x_i) \varphi_j(t \wedge T, dx) < \infty.$$

We have thus established the first three admissibility conditions, the remaining four being obvious. Since the map $t \mapsto t \wedge T$ is continuous, γ^* and, due to Condition 4, also β^* are continuous in t. Finally, Condition 4 and the continuity conditions for the original triplets imply weak continuity of

$$h_k(x) \varphi_j^*(t, dx) = h_k(x) \varphi_j(t \wedge T, dx) + h_k(x) x_i \varphi_j(t \wedge T, dx)$$

for $1 \leq k \leq m, k \neq j$, and of

$$h_k(x)^2 \varphi_j^*(t, dx) = h_k(x)^2 \varphi_j(t \wedge T, dx) + h_k(x)^2 x_i \varphi_j(t \wedge T, dx)$$

for $k \geq m + 1$ or $k = j$. Therefore $(\beta_j^*, \gamma_j^*, \varphi_j^*)$ are strongly admissible. $\qquad \square$

The next step is to work towards local absolute continuity of Q with respect to P. In view of [12, Lemma III.3.3], we do this by constructing a localizing sequence $(T_n)_{n \in \mathbb{N}}$ for M under P such that $T_n \uparrow \infty$ holds under Q as well. In the continuous case this can always be achieved by considering the hitting times $T_n = \inf\{t \in \mathbb{R}_+ : |M_t| \geq n\}$. This approach does not work in the presence of jumps, yet here a similar explicit construction is possible.

Lemma 3.6 *Let $(\beta_j(t), \gamma_j(t), \varphi_j(t))$, $j = 0, \ldots, d$, $t \in \mathbb{R}_+$ be strongly admissible Lévy-Khintchine triplets. Assume that a solution P to the corresponding affine martingale problem on $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ exists. Then the stopping times $(T_n)_{n \in \mathbb{N}}$ given by*

$$T_n = \inf\{t > 0 : |X_{t-}| \geq n \text{ or } |X_t| \geq n\}$$

satisfy $T_n \uparrow \infty$. If Condition 4 in Theorem 3.1 holds and $M = \mathscr{E}(X^i)^T$ is a local martingale for some $1 \leq i \leq d$ and $T \in \mathbb{R}_+$, then $(T_n)_{n \in \mathbb{N}}$ is a localizing sequence for M.

PROOF. $T_n \uparrow \infty$ follows immediately from the càdlàg property of X. Since $\mathscr{M}_{\mathrm{loc}}$ is stable under stopping, we know that $M^{T_n} \in \mathscr{M}_{\mathrm{loc}}$. By [12, I.1.47c] it remains to show that M^{T_n} is of class (D), i.e. $\{M_S^{T_n} : S \text{ finite stopping time}\}$ is uniformly integrable. It suffices to show

$$E \left(\sup_{t \in [0,T]} |M_{T_n \wedge t}| \right) < \infty \qquad (3.4)$$

because $M_t^{T_n}$ is constant for $t \geq T$. Let (B, C, ν) be the characteristics of M. By Lemma 2.3 the stopped process M^{T_n} admits the stopped characteristics $(B^{T_n}, C^{T_n}, \nu^{T_n})$. Since it is a local martingale, [12, II.2.38] yields its canonical decomposition

$$
\begin{aligned}
M^{T_n} &= M_0^{T_n} + (M^{T_n})^c + x * (\mu^{T_n} - \nu^{T_n}) \\
&= M_0^{T_n} + (M^{T_n})^c + (x 1_{\{|x| \leq 1\}}) * (\mu^{T_n} - \nu^{T_n}) + (x 1_{\{|x| > 1\}}) * (\mu^{T_n} - \nu^{T_n}).
\end{aligned}
$$

The definition of T_n and [12, I.4.61] yield

$$\sup_{t \in [0,T]} M_{t-}^{T_n} \leq \sup_{t \in [0,T]} \exp\left((X^i)_{t-}^{T_n}\right) \leq e^n. \qquad (3.5)$$

For the jump at t we obtain

$$\Delta M_t^{T_n} = \Delta \left(x 1_{\{|x| \leq 1\}} * (\mu^{T_n} - \nu^{T_n}) \right)_t + \Delta \left(x 1_{\{|x| > 1\}} * (\mu^{T_n} - \nu^{T_n}) \right)_t \qquad (3.6)$$

because $(M^{T_n})^c$ is continuous and $M_0^{T_n}$ is constant. By [12, II.1.27] we have

$$\sup_{t \in [0,T]} \Delta \left(x 1_{|x| \leq 1} * (\mu^{T_n} - \nu^{T_n}) \right)_t = \sup_{t \in [0,T]} \Delta M_t^{T_n} 1_{\{|\Delta M_t^{T_n}| \leq 1\}} \leq 1. \qquad (3.7)$$

Furthermore, we obtain

$$\sup_{t \in [0,T]} \Delta \left(x 1_{\{|x| > 1\}} * (\mu^{T_n} - \nu^{T_n}) \right)_t \leq \sum_{t \leq T} |\Delta M_t^{T_n}| 1_{\{|\Delta M_t^{T_n}| > 1\}} = |x| 1_{\{|x| > 1\}} * \mu_T^{T_n}.$$

By [12, II.1.8] we have

$$E \left(|x| 1_{\{|x| > 1\}} * \mu_T^{T_n} \right) = \int_0^{T_n \wedge T} \int_{\{|x| > 1\}} |x| F_t^M(dx) dt,$$

where F_t^M denotes the local Lévy measure of M in the sense of Definition 2.1. We can compute the differential characteristics of M through [14, Proposition 2]. With $G_t = \{x \in \mathbb{R}^d : M_{t-} |x_i| > 1\}$ and the definition of T_n this yields

$$
\begin{aligned}
\int_0^{T_n \wedge T} \int_{\{|x| > 1\}} |x| F_t^M(dx) dt &= \int_0^{T_n \wedge T} \int_{G_t} M_{t-} |x_i| \varphi_0(t, dx) dt \\
&\quad + \sum_{j=1}^m \int_0^{T_n \wedge T} \int_{G_t} M_{t-} |x_i| \varphi_j(t, dx) X_{t-}^j dt \\
&\leq n e^n \sum_{j=0}^m \int_0^{T_n \wedge T} \int_{\{|x_i| > \frac{1}{n}\}} |x_i| \varphi_j(t, dx) dt.
\end{aligned}
$$

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Since $|1/h_i|$ is bounded on $\{|x_i| > \frac{1}{n}\}$ and since it has a positive, bounded and continuous extension $\widetilde{h}$ to $\mathbb{R}^d$, it follows from Condition 4 in Theorem 3.1 that

$$\sup_{t \in [0,T]} \int_{\{|x_i| > \frac{1}{n}\}} |x_i| \varphi_j(t, dx) \leq \sup_{t \in [0,T]} \int_{\mathbb{R}^d} \widetilde{h}(x)|h_i(x)||x_i|\varphi_j(t, dx) < \infty$$

for $j = 0, \ldots, m$. Combining the above results yields

$$E\left(\sup_{t \in [0,T]} \Delta\left(x 1_{\{|x|>1\}} * (\mu^{T_n} - \nu^{T_n})\right)_t\right) < \infty. \tag{3.8}$$

In view of $M_t^{T_n} = M_{t-}^{T_n} + \Delta M_t^{T_n}$ and (3.5–3.8) we have that (3.4) holds as well. This proves the assertion. $\qquad\square$

Applying the previous result we get the following

Corollary 3.7 *Under the assumptions of Theorem 3.1, $(T_n)_{n \in \mathbb{N}}$ defined as in Lemma 3.6 is a localizing sequence for M under P and we have $T_n \uparrow \infty$, in particular Q-a.s.*

PROOF. M is a σ-martingale by Conditions 2 and 3 in Theorem 3.1 as derived above. Since it is nonnegative by Condition 2, it is a supermartingale and in particular a special semi-martingale. Hence it is a local martingale by [13, Corollary 3.1]. The claim then follows immediately from Condition 4 in Theorem 3.1 and from Lemmas 3.5 and 3.6. $\qquad\square$

Now we can prove that $Q|_{\mathscr{D}_T^0}$ is locally absolutely continuous with respect to $P|_{\mathscr{D}_T^0}$. Here, $\mathscr{D}_t^0$ denotes the σ-field generated by all maps $\alpha \mapsto \alpha(s)$, $s \leq t$ on $\mathbb{D}^d$. The filtration $(\mathscr{D}_t^0)_{t \in \mathbb{R}_+}$ is needed to apply [12, Theorem III.2.40].

Lemma 3.8 *Under the assumptions of Theorem 3.1 we have $Q|_{\mathscr{D}_T^0} \ll P|_{\mathscr{D}_T^0}$.*

PROOF. Since $M_0 = 1$, $M \geq 0$ and $(T_n)_{n \in \mathbb{N}}$ is a localizing sequence for $M \in \mathscr{M}_{\text{loc}}$ under P, we can define probability measures $Q^n \ll P$, $n \in \mathbb{N}$ with density processes M^{T_n}. We now show that the stopped canonical process $X^{T_n \wedge T}$ has differential characteristics $(b^* 1_{[\![0, T_n \wedge T]\!]}, c^* 1_{[\![0, T_n \wedge T]\!]}, F^* 1_{[\![0, T_n \wedge T]\!]})$ under both Q and Q^n, where (b^*, c^*, F^*) are defined in (2.1–2.3), (2.4–2.6) but relative to $(\beta_j^*, \gamma_j^*, \varphi_j^*)$ instead of $(\beta_j, \gamma_j, \varphi_j)$.

By construction and Lemma 2.3, $X^{T_n \wedge T}$ has the required characteristics under Q. Since $Q^n \ll P$, we can use [14, Proposition 4] to calculate the characteristics of $X^{T_n \wedge T}$ under Q^n. By $X^i \in \mathscr{M}_{\text{loc}}$ and [12, II.2.38] we have

$$X^i = X_0^i + e_i \bullet X^c + x_i * (\mu^X - \nu^X)$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ denotes the i-th unit vector. [14, Proposition 4] yields that $X^{T_n \wedge T}$ has the desired characteristics under Q^n as well.

The martingale problem corresponding to (b^*, c^*, F^*) and arbitrary initial law on $\mathbb{R}_+^m \times \mathbb{R}^{d-m}$ has a unique solution by Lemma 3.5. Since the solution process is Markovian and

by [12, Theorem III.2.40], local uniqueness in the sense of [12, III.2.37] is implied by uniqueness of the martingale problem. [12, VI.2.10] yields that the stopping times $T_n \wedge T$, $n \in \mathbb{N}$ are strict in the sense of [12, III.2.35]. Hence $Q^n|_{\mathscr{D}^0_{T_n \wedge T}} = Q|_{\mathscr{D}^0_{T_n \wedge T}}$. By construction we have $Q^n|_{\mathscr{D}^0_{T_n \wedge T}} \ll P|_{\mathscr{D}^0_{T_n \wedge T}}$, which implies $Q|_{\mathscr{D}^0_{T_n \wedge T}} \ll P|_{\mathscr{D}^0_{T_n \wedge T}}$. Let $A \in \mathscr{D}^0_T$ with $P(A) = 0$. From

$$A \cap \{T_n > T\} \in \mathscr{D}^0_{T_n} \cap \mathscr{D}^0_T = \mathscr{D}^0_{T_n \wedge T}$$

it follows that $Q(A \cap \{T_n > T\}) = 0$ for all $n \in \mathbb{N}$ and hence $Q(A) = 0$ by Corollary 3.7. This proves the claim. $\qquad\square$

If Q^n denotes the probability measure with density process M^{T_n} as in the proof of Lemma 3.8, we have $M_{T_n} = \frac{dQ^n}{dP}$. Since $M_{T_n} = M_{T_n \wedge T}$ is $\mathscr{D}^0_{T_n \wedge T}$-measurable, it is also the density on the smaller σ-field $\mathscr{D}^0_{T_n \wedge T}$, i.e. we have

$$M_{T_n} = \frac{dQ^n|_{\mathscr{D}^0_{T_n \wedge T}}}{dP|_{\mathscr{D}^0_{T_n \wedge T}}} = \frac{dQ|_{\mathscr{D}^0_{T_n \wedge T}}}{dP|_{\mathscr{D}^0_{T_n \wedge T}}} =: Z_n,$$

where the second equality is shown in the previous proof. Note that $(Z_n)_{n \in \mathbb{N}}$ is the martingale generated by $Z_\infty := dQ|_{\mathscr{D}^0_T}/dP|_{\mathscr{D}^0_T}$ on the discrete-time space $(\mathbb{D}^d, \mathscr{D}^0_T, (\mathscr{D}^0_{T_n \wedge T})_{n \in \mathbb{N}}, P)$. The martingale convergence theorem yields $M_{T_n} = Z_n \to Z_\infty$ a.s. for $n \to \infty$. Since we have $M_{T_n} = M_{T_n \wedge T} \to M_T$ a.s. for $n \to \infty$, this implies $M_T = Z_\infty$ a.s. and it follows that $E(M_T) = E(Z_\infty) = 1$, which proves Theorem 3.1.

3.3 Time-homogeneous exponentially affine martingales

We now apply the results of Section 3.1 to the homogeneous case. Throughout, let X^i with $1 \leq i \leq d$ be a component of an $\mathbb{R}^d$-valued semimartingale X admitting affine differential characteristics relative to admissible Lévy-Khintchine triplets $(\beta_j, \gamma_j, \varphi_j)$, $j = 0, ..., d$, which do not depend on t. Corollary 3.2 now reads as:

Corollary 3.9 *The process $\mathscr{E}(X^i)$ is a martingale if the following conditions hold:*

1. $\varphi_j(\{x \in \mathbb{R}^d : x_i < -1\}) = 0, \quad j = 0, \dots, m,$

2. $\int_{\{x_i > 1\}} x_i \varphi_j(dx) < \infty, \quad j = 0, \dots, m,$

3. $\beta_j^i + \int (x_i - h_i(x))\varphi_j(dx) = 0, \quad j = 0, \dots, d,$

4. $\int_{\{x_k > 1\}} x_k(1 + x_i)\varphi_j(dx) < \infty, \quad j, k = 1, \dots, m.$

Of course a counterpart to Corollary 3.4 can be derived similarly.

Example 3.10 Consider the stochastic volatility model of [2], which generalizes the model of [1] by allowing for jumps in the asset price X and in the volatility v:

$$X_t = X_0 + \mu t + L_{V_t} + \varrho Z_t,$$
$$dV_t = v_{t-}dt,$$
$$dv_t = -\lambda v_{t-}dt + dZ_t.$$

Here, μ, ϱ, λ are constants and L, Z denote independent Lévy processes with triplets (b^L, c^L, F^L) and $(b^Z, 0, F^Z)$, respectively. In addition, Z is supposed to be increasing. The affine structure of the differential characteristics of (v, X) can be calculated as in [14, Section 4.4]:

$$\beta_0 = \begin{pmatrix} b^Z \\ \mu + \varrho b^Z + \int (h(\varrho y) - \varrho h(y)) F^Z(dy) \end{pmatrix}, \quad \gamma_0 = 0,$$

$$\varphi_0(G) = \int 1_G(y, \varrho y) F^Z(dy) \quad \forall G \in \mathscr{B}^2,$$

$$\beta_1 = \begin{pmatrix} -\lambda \\ b^L \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 \\ 0 & c^L \end{pmatrix}, \quad \varphi_1(G) = \int 1_G(0, y) F^L(dy) \quad \forall G \in \mathscr{B}^2,$$

$$(\beta_2, \gamma_2, \varphi_2) = (0, 0, 0).$$

These triplets are admissible with $m = 1$. If moment conditions

$$\int_{\{\varrho y > 1\}} e^{\varrho y} F^Z(dy) < \infty, \quad \int_{\{y > 1\}} e^y F^L(dy) < \infty$$

and drift conditions

$$0 = \mu + \varrho b^Z + \int (e^{\varrho y} - 1 - \varrho h(y)) F^Z(dy), \quad 0 = b^L + \frac{1}{2}c^L + \int (e^y - 1 - h(y)) F^L(dy)$$

are satisfied, Corollary 3.4 yields that e^X is a martingale. These conditions are equivalent to e^L and $e^{\mu I + \varrho Z}$ being martingales, where I denotes the identity process $I_t = t$.

The following example shows that even in the homogeneous case with $\Delta X^i > -1$, Corollary 3.9 does not generally hold without the crucial moment condition 4.

Example 3.11 Let

$$(\beta_0, \gamma_0, \varphi_0) := (0, 0, 0),$$

$$\beta_1 := \begin{pmatrix} \frac{1}{2\sqrt{\pi}} \int_0^\infty h(y) y^{-\frac{3}{2}} (1+y)^{-1} dy \\ \frac{1}{2\sqrt{\pi}} \int_0^\infty (h(y) - y) y^{-\frac{3}{2}} (1+y)^{-1} dy \end{pmatrix}, \quad \gamma_1 := 0,$$

$$\varphi_1(G) := \frac{1}{2\sqrt{\pi}} \int_0^\infty 1_G(y, y) y^{-\frac{3}{2}} (1+y)^{-1} dy,$$

$$(\beta_2, \gamma_2, \varphi_2) := (0, 0, 0).$$

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This defines admissible Lévy-Khintchine triplets on $\mathbb{R}^2$ satisfying (2.7), but violating Condition 4 in Corollary 3.9 for $i = 2$. By Theorem 2.6 there exists a probability measure P on $(\mathbb{D}^2, \mathscr{D}^2, \mathbf{D}^2)$ such that X is a semimartingale with affine differential characteristics relative to these triplets and $X_0 = (1, 1)$ P-almost surely. Computing the differential characteristics (b^M, c^M, F^M) of $M = \mathscr{E}(X^2)$ with [14, Proposition 2] yields $c^M = 0$ and

$$b^M = \int (h(x) - x) F^M(dx) \quad \text{and} \quad \int_{\{|x| > 1\}} |x| F^M(dx) < \infty.$$

By [13, Lemma 3.1] it follows that M is a positive local martingale. Now suppose M were a true martingale. In view of Lemma A.2 we could then define a probability measure $Q \overset{\text{loc}}{\ll} P$ with density process M. Since $M = \mathscr{E}(x_2 * (\mu^X - \nu^X))$, an application of [14, Proposition 4] yields the differential characteristics $\partial X^1 = (b, c, F)$ of X^1 under Q, namely

$$b_t = \int h(x) F_t(dx), \quad c_t = 0, \quad F_t(G) = \frac{X^1_{t-}}{2\sqrt{\pi}} \int_{G \cap (0, \infty)} x^{-\frac{3}{2}} dx \quad \forall G \in \mathscr{B}.$$

Hence X^1 coincides in law under Q with the process in [6, Example 9.3], which explodes in $[0, 1]$ with strictly positive probability. Since this contradicts $Q|_{\mathscr{D}^2_1} \ll P|_{\mathscr{D}^2_1}$, we conclude that $M = \mathscr{E}(X^2)$ is not a martingale.

Recall that Conditions 1–3 in Corollary 3.9 essentially mean that $\mathscr{E}(X^i)$ is a non-negative local martingale. Condition 4, on the other hand, is not needed for strong admissibility of $(\beta_j^*, \gamma_j^*, \varphi_j^*)$ in (3.1–3.3). Hence we know from [6, Theorem 2.7] that there exists a unique Markov process whose conditional characteristic function satisfies (2.8) with respect to $(\beta_j^*, \gamma_j^*, \varphi_j^*)$. But in order to ensure that it does not explode in finite time and hence is a semimartingale in the usual sense, we must also require this process to be conservative (cf. [6, Theorem 2.12]). To establish conservativeness, one generally has to resort to the sufficient but not necessary criteria in [6, Proposition 9.1 and Lemma 9.2], which is precisely what is done in the proof of Theorem 3.1.

3.4 Processes with independent increments

Instead of time-homogeneity we consider now deterministic characteristics. The following result slightly generalizes a parallel statement in the proof of [8, Proposition 4.4] by dropping the assumption of absolutely continuous characteristics. Hence we also incorporate processes with fixed times of discontinuity.

Proposition 3.12 *Let X be a semimartingale with independent increments (a PII in the sense of [12]) satisfying $\Delta X > -1$. Then $\mathscr{E}(X)$ is a martingale if and only if it is a local martingale.*

PROOF. For the proof of the nontrivial implication suppose that $\mathscr{E}(X)$ is a local martingale. Without loss of generality we can assume $X_0 = 0$. Denote the characteristics of X by

(B, C, ν). From $X \in \mathscr{M}_{\mathrm{loc}}$, [13, Lemma 3.1] and [12, II.5.2] it follows that there exists a PII Y with triplet (B^*, C^*, ν^*) given by

$$B_t^* = B_t + C_t + xh(x) * \nu_t, \quad C_t^* = C_t, \quad \nu^*(dt, dx) = (1 + x)\nu(dt, dx).$$

Its law is uniquely determined. We now choose Q equal to the law of Y and proceed almost literally as in the proof of Thereom 3.1: Lemma 3.8 is derived as above by using [12, III.3.24] or [13, Lemma 5.1] rather than [14, Proposition 4]. Moreover, the proof of Lemma 3.6 must be slightly modified. $\qquad\square$

4 Locally absolutely continuous change of measure

In the context of measure changes, Theorems 3.1 can be used to derive a sufficient condition for local absolute continuity of the law of an affine processes relative to another, similar to [12, IV.4.32] for processes with independent increments.

Theorem 4.1 *Let Y and Z be $\mathbb{R}^d$-valued semimartingales admitting affine differential characteristics relative to triplets $(\beta_j(t), \gamma_j(t), \varphi_j(t))$ and $(\widetilde{\beta}_j(t), \widetilde{\gamma}_j(t), \widetilde{\varphi}_j(t))$, $j = 0, \ldots, d$, $t \in \mathbb{R}_+$, which satisfy the conditions in Theorem 2.6. We have $P^Z \overset{\mathrm{loc}}{\ll} P^Y$ if there exist continuous functions $H : \mathbb{R}_+ \to \mathbb{R}^d$ and $W : \mathbb{R}_+ \times \mathbb{R}^d \to [0, \infty)$ such that, for $j = 0, \ldots, d$ and all $t \in \mathbb{R}_+$,*

1. *$\int_0^t \int (1 - \sqrt{W(s, x)})^2 \varphi_j(s, dx) ds < \infty$,*

2. *$\widetilde{\varphi}_j(t, G) = \int 1_G(x) W(t, x) \varphi_j(t, dx), \quad \forall G \in \mathscr{B}^d$,*

3. *$\int |h(x)(W(t, x) - 1)| \varphi_j(t, dx) < \infty$,*

4. *$\widetilde{\beta}_j(t) = \beta_j(t) + H_t^\top \gamma_j(t) + \int h(x)(W(t, x) - 1)\varphi_j(t, dx)$,*

5. *$\widetilde{\gamma}_j(t) = \gamma_j(t)$,*

6. *the measure $\chi(W(t, x) - 1)(W(t, x) - 1)\varphi_j(t, dx)$ is weakly continuous in t.*

PROOF. As before, we denote the canonical process by X. In view of the proof of [12, II.1.33d], Condition 1 implies that the measure in Condition 6 is finite. Condition 1 and [12, II.1.33] with the stopping times from Lemma 3.6 yield $W - 1 \in G_{\mathrm{loc}}(\mu^X)$ under P^Y. Since H is continuous, it follows that

$$N = H \bullet X^c + (W - 1) * (\mu^X - \nu^X)$$

is a well defined local martingale. The differential characteristics of (X, N) under P^Y are affine relative to

$$\widehat{\beta}_j(t) = \begin{pmatrix} \beta_j(t) \\ \int (\chi(W(t,x) - 1) - W(t,x) + 1)\varphi_j(t, dx) \end{pmatrix},$$

$$\widehat{\gamma}_j(t) = \begin{pmatrix} \gamma_j(t) & \gamma_j(t) H_t \\ H_t^\top \gamma_j(t) & H_t^\top \gamma_j(t) H_t \end{pmatrix},$$

$$\widehat{\varphi}_j(t, G) = \int 1_G(x, W(t,x) - 1)\varphi_j(t, dx), \quad \forall G \in \mathscr{B}^{d+1} \setminus \{0\}, \quad 0 \le j \le d,$$

$$(\widehat{\beta}_{d+1}, \widehat{\gamma}_{d+1}, \widehat{\varphi}_{d+1}) = 0.$$

These triplets are strongly admissible: the first seven admissibility conditions are obviously satisfied, the eighth follows from Condition 6, the weak continuity conditions for φ_j and the continuity of H. The ninth condition is clear and the last is again a consequence of Condition 6. Moreover, Conditions 1–5 in Thereom 3.1 hold for $i = d + 1$: Condition 4 in Theorem 3.1 is a consequence of the strong admissibility of $(\beta_j, \gamma_j, \varphi_j)$, $(\widetilde{\beta}_j, \widetilde{\gamma}_j, \widetilde{\varphi}_j)$ and the continuity of H. Condition 1 above implies Condition 2 in Theorem 3.1 and Condition 3 is obviously satisfied. Condition 5 in Theorem 3.1 holds by

$$\int_{\{x_k > 1\}} x_k(1 + x_{d+1})\widehat{\varphi}_j(t, dx) = \int_{\{x_k > 1\}} x_k W(t,x)\varphi_j(t, dx) = \int_{\{x_k > 1\}} x_k \widetilde{\varphi}_j(t, dx),$$

which is uniformly bounded on $[0, T]$ by Condition (2.7) in Theorem 2.6.

By Theorem 3.1 we have that $\mathscr{E}(N)$ is a martingale. Since it is positive, we can use it as a density process to define a probability measure $Q \overset{\text{loc}}{\ll} P^Y$ on $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d)$ (cf. Lemma A.2). By [14, Proposition 4] the differential characteristics of the canonical process under Q and P^Z coincide. Therefore Theorem 2.6 yields $Q = P^Z$, which proves the claim. $\qquad\square$

Conditions 1–5 also appear as necessary and sufficient conditions in [12, IV.4.32] in the case of PII. Our proof is based on the results of [9]. Since the latter are only formulated for continuous triplets, we require the additional continuity condition 6. This property holds in the time-homogeneous case. Consequently, the remaining conditions for each triplet coincide with those for Lévy processes in [12, IV.4.39] in this case, except for Assumption (2.7) in Theorem 2.6, which is an additional moment condition on the Lévy measures φ_j, $j = 1, \ldots, m$.

Corollary 4.2 *Let Y and Z be $\mathbb{R}^d$-valued semimartingales with affine differential characteristics relative to triplets $(\beta_j, \gamma_j, \varphi_j)$ and $(\widetilde{\beta}_j, \widetilde{\gamma}_j, \widetilde{\varphi}_j)$, $j = 0, \ldots, d$, respectively, which satisfy the conditions of Theorem 2.6. Suppose there exist $H \in \mathbb{R}^d$ and a Borel function $W : \mathbb{R}^d \to [0, \infty)$ such that, for $0 \le j \le d$, we have*

  1. $\int (1 - \sqrt{W(x)})^2 \varphi_j(dx) < \infty$,

  2. $\widetilde{\varphi}_j(G) = \int 1_G(x)W(x)\varphi_j(dx), \quad \forall G \in \mathscr{B}^d,$

3. $\int |h(x)(W(x) - 1)|\varphi_j(dx) < \infty,$

4. $\widetilde{\beta}_j = \beta_j + H^\top \gamma_j + \int h(x)(W(x) - 1)\varphi_j(dx),$

5. $\widetilde{\gamma}_j = \gamma_j.$

Then we have $P^Z \overset{\text{loc}}{\ll} P^Y.$

Similar results could be derived from [5, Theorem 2.4] applied to the affine case. Due to our heavy use of [9], we end up with continuity conditions in the time-inhomogeneous case, whereas [5] only require measurability and a certain uniform boundedness for H and W. However, our moment conditions are sometimes less restrictive than the corresponding criterion in [5, Remark 2.5]. We give an example arising from a practical application.

Example 4.3 As in Example 3.10, we consider the stochastic volatility model of [2]. From Corollary 4.2 with $H \in \mathbb{R}^2$, $W(x) = e^{H^\top x}$ we obtain that the distribution corresponding to the transformed triplets is locally equivalent to the original one if we have

$$\int_{\{|x|>1\}} e^{H^\top x} \varphi_j(dx) < \infty, \quad j = 0, 1.$$

For the application of [5], one needs the slightly stronger moment condition

$$\int_{\{|x|>1\}} (H^\top x) e^{H^\top x} \varphi_j(dx) < \infty, \quad j = 0, 1.$$

5 Exponential Moments

Let X be a semimartingale with affine differential characteristics relative to strongly admissible Lévy-Khintchine triplets $(\beta_j(t), \gamma_j(t), \varphi_j(t))$, $j = 0, \ldots, d$, $t \in \mathbb{R}_+$. In [6, Propositions 6.1 and 6.4] (respectively [9, Propositions 4.1 and 4.3] for the time-inhomogeneous case), it is shown that a solution to the generalized Riccati equations from Theorem 2.6 always exists for initial values $u \in \mathbb{C}_-^m \times i\mathbb{R}^{d-m}$. Theorem 2.16 in [6] then asserts that if there exists an analytic extension of this solution to an open convex set containing $p \in \mathbb{R}^d$, the exponential moment $E(\exp(p^\top X_T))$ can be obtained by inserting the value p into the formula for the characteristic function.

The existence of this extension, however, may be difficult to verify, even for models without jumps. Using the results from Section 3, we show that $E(\exp(p^\top X_T))$ or, more generally, $E(\exp(p^\top X_T)|\mathscr{F}_t)$ can typically be obtained by solving the generalized Riccati equations (2.9, 2.10) with initial value p.

Theorem 5.1 *Let* $p \in \mathbb{R}^d$ *and* $T \in \mathbb{R}_+$. *Suppose that* $\Psi^0 \in C^1([0, T], \mathbb{R})$ *and* $\Psi^{(1,\ldots,d)} = (\Psi^1, \ldots, \Psi^d) \in C^1([0, T], \mathbb{R}^d)$ *satisfy*

1. $\int_{\{|x|>1\}} e^{\Psi^{(1,\ldots,d)}(t)^\top x} \varphi_j(t, dx) < \infty, \quad j = 0, \ldots, d, \quad \forall t \in [0, T],$

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2. $\Psi^{(1,\ldots,d)}(T) = p, \quad \frac{d}{dt}\Psi^j(t) = -\psi_j(t, \Psi^{(1,\ldots,d)}(t)), \quad j = 1, \ldots, d,$

3. $\Psi^0(t) = \int_t^T \psi_0(s, \Psi^{(1,\ldots,d)}(s))ds, \quad \forall t \in [0, T],$

4. $E(\exp(\Psi^{(1,\ldots,d)}(0)^\top X_0)) < \infty,$

5. $\sup_{t\in[0,T]} \int_{\{x_k>1\}} x_k e^{\Psi^{(1,\ldots,d)}(t)^\top x} \varphi_j(t, dx) < \infty, \quad 1 \le j, k \le m.$

Then we have

$$E\big(e^{p^\top X_T}\big|\mathscr{F}_t\big) = \exp\big(\Psi^0(t) + \Psi^{(1,\ldots,d)}(t)^\top X_t\big), \quad \forall t \le T. \tag{5.1}$$

PROOF. By Condition 1 we have $\psi_j(t, \Psi^{(1,\ldots,d)}(t)) < \infty$ for all $t \in [0, T]$. Define

$$N_t := \Psi^0(t) + \Psi^{(1,\ldots,d)}(t)^\top X_t.$$

Since $\Psi^{(1,\ldots,d)}$ is continuously differentiable, all Ψ^j are of finite variation. Hence $[\Psi^j, X^j] = 0$ and

$$\begin{pmatrix} X - X_0 \\ N - N_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \Psi^{(1,\ldots,d)}(I) & \frac{d}{dt}\Psi^0(I) + X^\top \frac{d}{dt}\Psi^{(1,\ldots,d)}(I) \end{pmatrix} \bullet \begin{pmatrix} X \\ I \end{pmatrix}$$

by the fundamental theorem of calculus and partial integration in the sense of [12, I.4.45]. From this representation we obtain the differential characteristics $\partial(X, N)$ by using [14, Propostion 2]. They are affine relative to time-inhomogeneous triplets $(\hat{\beta}_j, \hat{\gamma}_j, \hat{\varphi}_j)$ given by

$$\hat{\beta}_j(t) = \begin{pmatrix} \beta_j(t) \\ \frac{d}{dt}\Psi^j(t) + \Psi^{(1,\ldots,d)}(t)^\top \beta_j(t) + \\ + \int (h(\Psi^{(1,\ldots,d)}(t)^\top x) - \Psi^{(1,\ldots,d)}(t)^\top h(x))\varphi_j(t, dx) \end{pmatrix},$$

$$\hat{\gamma}_j(t) = \begin{pmatrix} \gamma_j(t) & \gamma_j(t)\Psi^{(1,\ldots,d)}(t) \\ \Psi^{(1,\ldots,d)}(t)^\top \gamma_j(t) & \Psi^{(1,\ldots,d)}(t)^\top \gamma_j(t)\Psi^{(1,\ldots,d)}(t) \end{pmatrix},$$

$$\hat{\varphi}_j(t, G) = \int 1_G(x, \Psi^{(1,\ldots,d)}(t)^\top x)\varphi_j(t, dx), \quad \forall G \in \mathscr{B}^{d+1}$$

for $j = 0, \ldots, d$ and

$$(\hat{\beta}_{d+1}, \hat{\gamma}_{d+1}, \hat{\varphi}_{d+1}) = (0, 0, 0).$$

From admissibility of the original triplets $(\beta_j, \gamma_j, \varphi_j)$ and continuity of Ψ^j, $j = 0, \ldots, d$, we infer that $(\hat{\beta}_j, \hat{\gamma}_j, \hat{\varphi}_j)$ are strongly admissible. The prerequisites of Corollary 3.4 are satisfied for $i = d + 1$: the first follows immediately from Condition 4. The second is a consequence of Condition 1 and the fact that all φ_j are Lévy measures, while the third follows from the definition of Ψ^0, $\Psi^{(1,\ldots,d)}$. The fourth prerequisite of Corollary 3.4 follows again from the continuity of $\Psi^{(1,\ldots,d)}$ while the fifth is just Condition 5. Therefore $\exp(N^T)$ is a martingale. For $t \le T$ the martingale property yields

$$E(e^{p^\top X_T}|\mathscr{F}_t) = E(\exp(N_T)|\mathscr{F}_t) = \exp(N_t) = \exp(\Psi^0(t) + \Psi^{(1,\ldots,d)}(t)^\top X_t),$$

which proves the claim. $\qquad\qquad\qquad\square$

Condition 1 is only needed for the ordinary differential equation in Condition 2 to be defined. It is automatically satisfied if the Lévy measures φ_j have compact support, i.e. if X has bounded jumps. Condition 2 and 3 mean that Ψ^0 and $\Psi^{(1,\dots,d)}$ solve equations (2.9, 2.10) with initial value p. In the common situation that X_0 is deterministic, Condition 4 obviously holds. The moment condition 5 is crucial. It holds e.g. if the Lévy measures φ_j have compact support or if $\varphi_1, \dots, \varphi_m$ are concentrated on the set $\{x \in \mathbb{R}^d : x_1 = \dots = x_m = 0\}$. This is the case for many affine stochastic volatility models as e.g. the time-changed Lévy models proposed by [2]. The proof of Theorem 5.1 shows that the theory of time-inhomogeneous affine processes can become useful even in the study of time-homogeneous processes.

A   Appendix

In this appendix we state a time-inhomogeneous version of [6, Lemma 9.2], i.e. a sufficient criterion for an affine Markov process to be conservative. Moreover, we recall a statement on the existence of probability measures on the Skorohod space which are defined in terms of their density process.

Let $(\beta_j, \gamma_j, \varphi_j)$, $j = 0, \dots, d$, be strongly admissible Lévy-Khintchine triplets in the sense of Definition 2.4. Then by [9, Theorem 2.13] there exists a unique Markov process with state space $D = \mathbb{R}^m \times \mathbb{R}^{d-m}$ and transition function $(p_{t,T}(x, d\xi))_{t \leq T < \infty}$ such that

$$\int_{D \setminus \{0\}} f_u(\xi) p_{t,T}(x, d\xi) = \exp(\Psi^0(t, T, u) + \Psi^{(1,\dots,d)}(t, T, u)^\top x), \quad x \in D, \qquad (A.1)$$

where $f_u(x) = \exp(u^\top x)$ for $u \in i\mathbb{R}^d$ and the mappings Ψ^0, $\Psi^{(1,\dots,d)} = (\Psi^1, \dots, \Psi^d)$ are given as the unique solutions to the generalized Riccati equations (2.9) and (2.10).

By [9, Theorem 2.14], this Markov process is a semimartingale in the usual sense and the unique solution to the affine martingale problem corresponding to $(\beta_j, \gamma_j, \varphi_j)$, $j = 0, \dots, d$, if it is *conservative*, i.e. if $p_{t,T}(x, D) = 1$ for all $0 \leq t \leq T < \infty$ and $x \in D$. In view of (A.1), this is equivalent to $\Psi^0(t, T, 0) = 0$ and $\Psi^{(1,\dots,d)}(t, T, 0) = 0$ for all $0 \leq t \leq T < \infty$. A sufficient condition, which extends [6, Lemma 9.2] to the time-inhomogeneous case, is provided in the following

Lemma A.1 *Let $(\beta_j, \gamma_j, \varphi_j)$, $j = 0, \dots, d$, be strongly admissible Lévy-Khintchine triplets. Then if*

$$\sup_{t \in [0,T]} \int_{\{x_k > 1\}} x_k \varphi_j(t, dx) < \infty, \quad \text{for } j, k = 1, \dots, m, \qquad (A.2)$$

the corresponding affine Markov process is conservative.

PROOF. The proof is a modification of Lemma 9.2 and the first part of Lemma 9.1 in [6]. Let

$$\mathbb{R}_-^m := \{x \in \mathbb{R}^m : \text{Re}(x_i) \leq 0, \ \forall i\}, \quad \mathbb{R}_{--}^m := \{x \in \mathbb{R}^m : \text{Re}(x_i) < 0, \ \forall i\}.$$

Apparently, the function $g = 0$ is an $\mathbb{R}^m_-$-valued solution of the initial value problem

$$\frac{\partial}{\partial t} g(t) = \psi^{(1,\dots,m)}(T - t, (g(t), 0)), \quad g(0) = 0, \tag{A.3}$$

where $\psi^{(1,\dots,m)} := (\psi_1, \dots, \psi_m)$. In view of [9, Theorem 2.13], $\Psi^{(1,\dots,m)}(T - \cdot, T, 0) := (\Psi^1(T - \cdot, T, 0), \dots, \Psi^m(T - \cdot, T, 0))$ also solves (A.3) on $[0, T]$. From [9, Proposition 4.1] it follows that $\Psi^{(1,\dots,m)}(T - \cdot, T, (v, 0))$ is $\mathbb{R}^m_{--}$-valued for $v \in \mathbb{R}^m_{--}$. Therefore it is $\mathbb{R}^m_-$-valued for $v \in \mathbb{R}^m_-$ by [9, Lemma 3.1 and Proposition 4.3]. Similarly as in [6, Lemma 5.3] it now follows from (A.2) that $\psi^{(1,\dots,m)}(t, (v, 0))$ is locally Lipschitz continuous in $v \in \mathbb{R}^m_-$. Hence 0 is the unique $\mathbb{R}^m_-$-valued solution to (A.3) and it follows that $\Psi^{(1,\dots,m)}(t, T, 0) = 0$ for $t \in [0, T]$ and hence $\Psi^{((1,\dots,d)}(t, T, 0) = 0$ for $t \in [0, T]$ by (2.10). In view of (2.9) this implies $\Psi^0(t, T, 0) = 0$ for $t \leq T$, which proves the assertion. $\qquad\square$

Lemma A.2 *Let* $(\mathbb{D}^d, \mathscr{D}^d, \mathbf{D}^d, P)$ *denote the Skorohod space of càdlàg functions endowed with some probability measure* P *and* Z *some nonnegative martingale on that space with* $E(Z_0) = 1$. *Then there exists a probability measure* $Q \overset{\mathrm{loc}}{\ll} P$ *with density process* Z.

PROOF. For any $t \in \mathbb{R}_+$ there exists a probability measure Q_t on $\mathscr{D}^d_t$ with density Z_t. The family $(Q_t)_{t \in \mathbb{R}_+}$ is consistent in the sense that $Q_t|_{\mathscr{D}^d_s} = Q_s$ for $s \leq t$. The assertion now follows from [16, Theorem V.4.1] by using that $(\mathbb{D}^d, \mathscr{D}^d_t)$ is a standard Borel space, since it is isomorphic to the Skorokhod space $\mathbb{D}^d([0, t])$ of $\mathbb{R}^d$-valued càdlàg functions on $[0, t]$ equipped with its Borel σ-algebra, which is a standard Borel space by e.g. [16, Theorem VII.6.3]. $\qquad\square$

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