Q1. Prove or disprove that a graph is bipartite if and only if no two adjacent vertices have the same distance from any other vertex.

Q2. Prove or disprove that every connected graph contains a walk that traverses each of its edges exactly once in each direction.

Q3. Let $A = (a_{ij})_{n\times n}$ be the adjacency matrix of the graph G. Show that the matrix $A^k = (a'_{ij})_{n\times n}$ displays, for all $i, j \leq n$, the number a'_{ij} of walks of length k from v_i to v_j in G.

Q4. Let G be a connected graph whose edges have been assigned real numbers. As mentioned on page 14 of the text, G has at least one spanning tree. The *weight* of a spanning tree is the sum of the numbers on its edges. The *spectrum* of a spanning tree is the list of the numbers on its edges (each number listed as many times as it occurs on the edges of the tree) in non-decreasing order. Show that any two spanning trees of minimum weight (among all spanning trees of G) must have the same spectrum.

Q5. An oriented complete graph is called a *tournament*. The *outdegree* of a vertex v, written $od(v)$, is the number of edges directed away from v. Let T be a tournament with n vertices. Find a formula for the number of directed 3-cycles in T in terms of n and the outdegrees of the vertices of T.

Q6. Let G be a graph with vertices $\{v_1, v_2, \ldots, v_n\}$. Let the matrix M be defined by

$$m_{ij} = \begin{cases} d(v_i) & \text{if } i = j \\ -1 & \text{if } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

The **Matrix Tree Theorem** states that the number of spanning trees of G is equal to the value of any cofactor of M. Use the matrix tree theorem to find the number of spanning trees in $K_{n,n}$.