Complete the Square

Prove the following for all real numbers \( x, y \) such that \( x \geq y \),

\[
2x^2 + y^2 \geq 2xy + x + y - 1
\]

\[
\iff x^2 + x^2 + y^2 \geq 2xy + x + y - 1
\]

\[
\iff x^2 - 2xy + y^2 + x - x - y + 1 \geq 0
\]

\[
\iff (x - y)^2 + x^2 - x - y + 1 \geq 0
\]

\[
\iff (x - y)^2 + x^2 - x - x + x - y + 1 \geq 0
\]

\[
\iff (x - y)^2 + x^2 - 2x + 1 + x - y \geq 0
\]

\[
\iff (x - y)^2 + (x - 1)^2 + (x - y) \geq 0
\]

We know \( (x - y)^2 \geq 0 \), \( (x - 1)^2 \geq 0 \)

and since \( x \geq y \), \( x - y \geq 0 \)

So sum of nonnegative is nonnegative.
Induction

1. Remember Fibonacci Numbers?

\[ F_n = \begin{cases} 
0 & \text{if } n = 0; \\
1 & \text{if } n = 1; \\
F_{n-1} + F_{n-2} & \text{if } n > 1.
\end{cases} \]

Now, prove for all natural number \( n \),

\[ \sum_{i=0}^{n} iF_i = nF_{n+2} - F_{n+3} + 2 \]

Let \( P(n) = \sum_{i=0}^{n} iF_i = nF_{n+2} - F_{n+3} + 2 \).

**Base** \( P(1) \) true b/c \( 1F_1 = 1 = (1)(F_3) - F_4 + 2 = 2 - 3 + 2 = 1 \)

**1.H.** Assume \( P(k) \) true for some \( k \in \mathbb{N} \).

**1.S.** By 1.H.

\[ \sum_{i=0}^{k} iF_i = kF_{k+2} - F_{k+3} + 2 \]

\[ \Rightarrow \sum_{i=0}^{k} iF_i + (k+1)(F_{k+1}) = kF_{k+2} - F_{k+3} + 2 + (k+1)(F_{k+1}) \]

\[ = k(F_{k+3} - F_{k+1}) - (F_{k+4} - F_{k+2}) + 2 + kF_{k+1} + F_{k+1} \]

\[ = kF_{k+3} - F_{k+4} + F_{k+2} + F_{k+1} + 2 \]

\[ = kF_{k+3} - F_{k+4} + F_{k+2} + F_{k+1} + 2 \]

\[ = (k+1)(F_{k+3}) - F_{k+4} + 2 \]
Function

1. Prove that the following function \( f : \mathbb{R} \to \mathbb{R} \) is bijective.

\[
f(x) = (3(x+1)^3 + 5)^3
\]

Claim: \( g : \mathbb{R} \to \mathbb{R} \) by \( g(x) = x^3 \) is bijective.

\[
\begin{align*}
\text{INJ} & \quad \text{Let } x, y \in \mathbb{R}, \\
\text{Assume } g(x) &= g(y) \\
\Rightarrow \quad x^3 &= y^3 \\
\Rightarrow \quad x &= y
\end{align*}
\]

\[
\begin{align*}
\text{SURJ} & \quad \text{Let } b \in \mathbb{R}, \\
\text{Then take } \sqrt[3]{b} & \in \mathbb{R} \\
g(\sqrt[3]{b}) &= (\sqrt[3]{b})^3 = b
\end{align*}
\]

Claim: \( h : \mathbb{R} \to \mathbb{R} \) by \( h(x) = ax + b \) for real numbers, where \( a \neq 0 \) is bijective.

\[
\begin{align*}
\text{INJ} & \quad \text{Let } x, y \in \mathbb{R}, \\
\text{Assume } h(x) &= h(y) \\
ax + b &= ay + b \\
\Rightarrow \quad x &= y
\end{align*}
\]

\[
\begin{align*}
\text{SURJ} & \quad \text{Let } c \in \mathbb{R}, \\
\text{Take } c - \frac{b}{a} & \in \mathbb{R} \\
h\left( c - \frac{b}{a} \right) &= a \left( c - \frac{b}{a} \right) + b = c
\end{align*}
\]

Thus \( h'(x) = x + 1 \), \( h''(x) = 3x + 5 \) is bijective.

So \( f(x) = g(h''(g(h'(x)))) \)

Since \( f \) is a composition of bijections, \( f \) is bijective.
Counting 2 Ways

Q. Prove the following by counting 2 ways.

\[ \sum_{i=0}^{n} \binom{n}{i} 100^i 101^{n-i} = 100^n \]

Let \( S \) = the set of ways to assign grades
(from 0 to 100) to \( n \) concepts students.

Clearly, \( |S| = 101^n \), so RHS counts \( S \).

Let \( S_c \) = the set of ways to assign grades
(from 0 to 100) to \( n \) concepts students
such that \( n-c \) students got perfect.

To form \( S_c \):
1) choose \( n-c \) students who got perfect
\[ \binom{n}{n-c} = \binom{n}{c} \]

2) For remaining \( c \) students, assign
grades 0 - 99
\[ \Rightarrow 100^c \]

Clearly, \( S_0, \ldots, S_n \) partition \( S \).

Thus
\[ |S| = \sum_{i=0}^{n} |S_i| = \sum_{i=0}^{n} \binom{n}{i} (100^i) \]
so LHS counts \( S \).
What's the probability of a 5 card poker hand that has the following properties?

- Has three cards of the same rank.
- Includes all four suits.
- Includes three different ranks.

Steps:

1) Choose a rank. \( \binom{13}{1} \)

2) Choose 3 suits. \( \binom{4}{3} \)

3) Choose 2 other ranks. \( \binom{12}{2} \)

4) Of \( 4^2 \) ways to choose suits for the last two cards, \( 3^2 \) of them do not include unused suit.

So \( \binom{7}{7} \) ways.

\[
\begin{pmatrix}
\binom{13}{1} & \binom{4}{3} & \binom{12}{2} & \binom{7}{7} \\
1 & 5 & 2 & 1
\end{pmatrix}
\]

\( \binom{52}{5} \)

\( \text{total # poker hand} \)
Expectation

There are 8 questions on a concepts final exam. Suppose that you had no idea how to solve any of them. So your strategy for the exam was as follows:

1. Choose a problem at random.
2. Stare at it for 1 minute. Then repeat step 1.

What's the expected number of minutes until you will stare at all the questions?

\[ X = \text{the number of minutes until you will stare at all the questions.} \]

\[ X_i = \text{The number of minutes until you stare at a new question when } i \text{ question remain unstared.} \]

Then \( X = \sum_{i=1}^{8} X_i \)

Note: The probability of staring at a new question when \( i \) question remain unstared is \( \frac{8-i}{8} \).

by Prop 9.29, the expected number of trials \( \frac{8}{c} \).

So, \( E[X] = \sum_{i=1}^{8} E[X_i] = \sum_{i=1}^{8} \frac{8}{c} \)

\[ = \frac{8}{1} + \frac{8}{2} + \frac{8}{3} + \frac{8}{4} + \frac{8}{5} + \frac{8}{6} + \frac{8}{7} + \frac{8}{8} \]

\[ = 8 + \frac{8}{3} + \frac{4}{1} + \frac{8}{5} + \frac{4}{3} + \frac{2}{7} + 1 = 19 + \frac{96}{35} \]