More Counting 2 Ways

Prove this by counting two ways:

\[ \sum_{i=1}^{n} i = \binom{n+1}{2} \]

Note that the right hand side simplifies to something we’ve seen before:

\[ \binom{n+1}{2} = \frac{(n+1)!}{2!(n+1-2)!} = \frac{(n+1)(n)}{2} \]

Again, on exam, follow these steps to count a set in two ways:

1.) Choose easy side to count.
2.) Define set \( S \) you’re counting looking at that side.
   a. It can be “the set of \( k \) element subset of \([n]\)” for \( \binom{n}{k} \), or
   b. It can be “the set of ways to choose \( k \) XXX’s from the set of \( n \) XXX’s.”
3.) Count the easy side.
4.) Count the hard side. Partition \( S \) into \( S_i \) if necessary. In that case, invoke rule of sum at the end.
5.) Don’t need to prove that the partition is indeed a partition, but explain briefly in English.

**Proof.**

We will count the right hand side.

Let \( S \) be the set of 2 element subset of \([n+1]\).

By definition, \( |S| = \binom{n+1}{2} \).

Since there’s a summation, we’ll do partitioning. Since there are 2 elements in each set in \( S \), we have a choice to partition based on the smaller element or the larger element.

Let \( S_i \) be the set of 2 element subset of \([n+1]\) where the largest element in each set is \( i+1 \).

Clearly, since we’re choosing the largest elements and the largest element is in between 2 and \( n+1 \),

\[ S_1, S_2, \ldots, S_n \]

partition \( S \). Furthermore, \( S_i \) can be formed as follows:

Step 1.) Pick the largest element.
Step 2.) Pick the smaller element (\( i \) choices).

Thus, \( |S_i| = \binom{i}{1} = i \)

And by rule of sum, \( |S| = \sum_{i=1}^{n} |S_i| = \sum_{i=1}^{n} i \).

Since LHS and RHS both count \( S \), they are equal.
3. Prove the following by counting 2 ways when \( q \) is an integer greater than 1. This is a geometric sum formula.

\[
\sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1}
\]

\[
\Rightarrow (q-1) \sum_{i=0}^{n-1} q^i = q^n - 1
\]

\[
\Rightarrow 1 + (q-1) \sum_{i=0}^{n-1} q^i = q^n
\]

- Let \( S \) be the set of ways to assign \( q \) colors to \([n]\), where one of the colors is red.
- By \( n \) step process w/ each \( q \) choices, \( |S| = q^n = \text{RHS} \).

- Now, partition \( S \) into the following sets:
  
  \( S_B = \) the set of ways to assign \( q \) colors to \([n]\), such that all are colored red.
  
  \( S_{i,c} = \) the set of ways to assign \( q \) colors to \([n]\), such that the maximum non-red element is \( i+1 \) with color \( c \).

  To form \( S_{i,c} \):
  1) Color \( i+1 \)th element as \( c \), \( \rightarrow 1 \) way
  2) For all other \( i \) elements, \( \rightarrow q^i \) ways

  Then \( S_{i,c} \) where \( 0 \leq i \leq n-1 \) and \( c \neq \text{red} \)
  and \( S_B \) partition \( S \).

Thus \( |S| = 1 + (q-1) \sum_{i=0}^{n-1} q^i \), so LHS counts \( S \).
4. Prove the following by counting 2 ways.

\[ \sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n} \]

Let \( S \) be the set of \( n \) element subset of \([x+n+1]\). Clearly, RHS counts this.

Let \( S_k \) be the set of \( n \) element subset of \([x+n+1]\) such that the smallest number that's not in the subset is \( n-k+1 \).

For example, \([1, 2, 3, 5, 6, 8, \ldots]\)

If this is an \( n \) element subset of \([x+n+1]\), since \( 4 \) is the smallest number that's not in the subset, this will go to \( S_{n-3} \) because \( 4 = n - k + 1 \)

\[ \Rightarrow k = n - 3 \]

The smallest missing number possible is \( 1 \), so \( 1 \leq n - k + 1 \)

and the largest missing number possible is \( n+1 \) (because then \( \{1, \ldots, n\} \) is in it, which is \( n+1 \) element subset), so \( n+1 \leq n - k + 1 \)

\[ \Rightarrow k \leq n \]

Thus \( S_0, \ldots, S_{n-k} \) partition \( S \), and \( S_k \) can be formed by

1) Put \( n-k \) elements in set

2) From \( \{n-k+2, \ldots, x+n+1\} \) choose \( k \) more elements,

Step (1) has 1 way and (2) has \( \binom{x+k}{k} \) ways. So \( |S| = \sum_{k=0}^{n} |S_k| = \sum_{k=0}^{n} \binom{x+k}{k} \) ·