SOLUTIONS TO HW 9

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II.13: N becomes an A-module via the scalar multiplication $an = f(a)n$. Consider the map from $B \times N$ to N given by scalar multiplication. This is A-bilinear, because $a(bn) = f(a)(bn) = (f(a)b)n = b(f(a)n)$ [remember that the A-module structures of B and N are defined in terms of f]

By the universal property of tensor product, we get an A-linear map $p : N_B \to N$ such that $p(b \otimes n) = bn$. This map is actually B-linear because $p(c(b \otimes n)) = p(cb \otimes n) = cbn = cp(b \otimes n)$.

The map g is A-linear and $p(g(y)) = p(1 \otimes y) = y$, so easily g is injective. What is more if $x \in N_B$ then $x = (x - g(p(x))) + g(p(x))$, and $p(x - g(p(x))) = p(x) - p(x) = 0$. So x is the sum of an element of $ker(p)$ and an elemnt of $im(g)$. Finally if $g(y) \in ker(p)$ then $y = p(g(y)) = 0$, so the intersection of $im(g)$ and $ker(p)$ is trivial.

III.4: Attempt to define a map from $S^{-1}B$ to $T^{-1}B$ by $b/s \mapsto b/f(s)$. It is clear that if this map is well-defined it is surjective.

If $b/s = b'/s'$ then by definition $u(s'b - sb') = 0$ for some $u \in S$ (careful: in this equation we are thinking of B as an A-module). By the defn of the A-module structure for B, this amounts to saying that in B we have $f(u)(f(s')b - f(s)b') = 0$, and since $f(u) \in T$ we have $b/f(s) = b'/f(s')$. A similar argument shows that map is injective.

Since B is an A-module, $S^{-1}B$ has a structure as an $S^{-1}A$-module with scalar multiplication given by $a/sb/s' = f(a)b/ss'$. On the other hand the composition $a \mapsto f(a) \mapsto f(a)/1$ of f and the natural maps from B to $T^{-1}B$ takes all elements of S to units, so there is a HM from $S^{-1}A$ to $T^{-1}B$ given by $a/s \mapsto f(a)/f(s)$; this induces an $S^{-1}A$-module structure on $T^{-1}B$ as usual.

We have an isomorphism of $S^{-1}A$-modules because $a/sb/s' = f(a)b/ss' \mapsto f(a)b/f(s)f(s') = a/sb/f(s')$.

III.7 : Let the complement of S be the union of a family X of prime ideals. For each prime ideal $P \in X$ we have $xy \in P$ iff $x \in P$ or $y \in P$, so $xy \in S$ iff $xy \notin P$ for all $P \in X$ iff $x \notin P$ and $y \notin P$ for all $P \in X$ iff $x \in S$ and $y \in S$.

Conversely let S be saturated and let $a \notin S$. Since S is saturated the ideal (a) is disjoint from S. Let P be maximal among ideals containing (a) and disjoint from S; by the usual argument P is prime. This shows $A \setminus S$ is a union of prime ideals.

Now let S be multiplicatively closed and let U be the union of the prime ideals which do not meet S. Let $\bar{S}$ be the complement of U, so that clearly $\bar{S}$ contains S and $\bar{S}$ is saturated. Suppose that T is any saturated set that contains S. The complement of T is a union of prime ideals which all avoid S, so is a subset of U: therefore $\bar{S}$ is a subset of T and $\bar{S}$ is the least saturated set containing S.

If $S = 1 + I$ for ideal I, let P be prime and disjoint from $1 + I$. This is equivalent to $P + I$ being proper, in which case there is a maximal M with $P + I \subseteq M$. So the union of the prime ideals avoiding S is the union of the maximal ideals containing

I; now a is in such an ideal iff $a + I$ is in some maximal ideal of A/I iff $a + I$ is not a unit in A/I. So the saturation is the set of a with $a + I$ a unit in R/I.

III.9: Let P be a minimal prime ideal, so that by Exercise 6 the complement S of P is maximal among mc sets not containing zero. Let $x \notin S$, and by maximality choose $n > 0$ minimal such that $0 \in x^n S$. If $0 = x^n s$ for $s \in S$ then $x(x^{n-1}s) = 0$ and $x^{n-1}s \neq 0$, so x is a zero-divisor and $x \in D$.

Given an mc set S, the kernel of the map $a \mapsto a/1$ consists of those a such that $as = 0$ for some $s \in S$. So the kernel = zero iff $as \neq 0$ for all $a \neq 0$ and all $s \in S$ iff $S \subseteq S_0$.

Let $b/s \in S_0^{-1}A$. If $b \in S_0$ this is a unit because $s/b \in S_0^{-1}A$. If $b \notin S_0$ then $b \in D$, so find $c \neq 0$ with $bc = 0$. Since $cd \neq 0$ for all $d \in S_0$, $c/1 \neq 0$ and we have $b/s \times c/1 = bc/s = 0$.

Let A be a ring in which nonunits are zero-divisors and consider b/s where s is not a zero-divisor. s is a unit so $st = 1$. By the definition of equality $b/s = bt/1$, so the map $a \mapsto a/1$ is surjective.

VII:10: generalise the proof of the Basissatz.