

HW 7

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Exercise from text: there are various ways to do this one. Avi and Paul pointed out that the functor “tensor product with a fixed M ” is a left adjoint so commutes with colimits.

By hand we can do it as follows. If we consider the map $M \times (N_1 \oplus N_2)$ to $(M \otimes N_1) \oplus (M \otimes N_2)$ given by $(m, (n_1, n_2)) \mapsto (m \otimes n_1, m \otimes n_2)$ it is bilinear, so there is a unique map ϕ from $M \times (N_1 \oplus N_2)$ to $(M \otimes N_1) \oplus (M \otimes N_2)$ such that $m \otimes (n_1, n_2) \mapsto (m \otimes n_1, m \otimes n_2)$. If now ψ is a bilinear map from $M \times (N_1 \oplus N_2)$ to some O then $\psi(m, (n_1, n_2)) = \psi(m, (n_1, 0)) + \psi(m, (0, n_2))$. Each of the maps $(m, n_1) \mapsto \psi(m, (n_1, 0))$ and $(m, n_2) \mapsto \psi(m, (0, n_2))$ is bilinear and factors uniquely through the relevant tensor product, and it follows that ψ factors uniquely through ϕ .

II.1 Since m and n are coprime there exist integers a and b such that $am + bn = 1$. So $1 \otimes 0 = (am + bn) \otimes 0 = m \otimes a + b \otimes n = 0$, and similarly $0 \otimes 1 = 0$.

II.6 It is routine to check that $M[x]$ is an $A[x]$ module. For the other part we show scalar multiplication restricted to $A[x] \times M$ is a universal A -bilinear map; to see this let ϕ be an A -bilinear map to some N , define $\psi : M[x] \rightarrow N$ by $\psi(\sum_i m_i x^i) = \sum_i \phi(x^i, m_i)$, and check it works. By the usual argument this universal property implies there is a unique IM from $A[x] \times M$ to $M[x]$ mapping $f \otimes m$ to fm .

Left over from HW6. An easy example of a projective module which is not free. Consider $M = \{0, 3\}$ as a $\mathbb{Z}/6\mathbb{Z}$ -submodule of $\mathbb{Z}/6\mathbb{Z}$. It is obviously not free, but we claim it is projective. To see this let f be a map from M to B and let g be any surjective map from A to B . Let $f(3) = z$ and note that $2z = 0$. Define a map h from M to A by $h(3) = 3w$ for any w such that $g(w) = z$. h is a module HM because $3w + 3w = 0$, and also $g(h(3)) = g(3w) = 3g(w) = 3f(3) = f(9) = f(3)$.

Cultural note: how did I know this would work? Well, it is not hard to see that a direct summand of a projective module is projective while a direct summand of a free module need not be free.