SOLUTIONS TO HW 5 WITH COMMENTS

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1: Let α be an algebraic integer and suppose first that α is rational, say $\alpha = p/q$ where p and q are coprime and $q > 0$. Let $f = x^n + a_{n_1} x^{n-1} + \ldots + a_0 \in \mathbb{Z}[x]$ be such that $f(\alpha) = 0$, and rearrange to get an equation $p^n = -a_{n_1} p^{n_1} q - \ldots - a_0 q^n$. Clearly any prime factor of q will divide p, so there cannot be any such factors and $q = 1$.

Now let α be an arbitrary algebraic integer with $f(\alpha) = 0$ for some monic $f \in \mathbb{Z}[x]$. Let m be the minimal polynomial of α over $\mathbb{Q}$, so that m divides f in $\mathbb{Q}[x]$, say $f = mg$. By Gauss' lemma $m \in \mathbb{Z}[x]$.

Alternative proof (more high tech): m is the product of terms $x - \alpha_i$ where the α_i are the conjugates of α. Each α_i is a root of f hence an algebraic integer. The algebraic integers form a ring so the coefficients of m are algeberaic integers in $\mathbb{Q}$, hence are integers.

2: It is important to keep in mind that the units of $k[x, y]$ are the nonzero members of k while the units of $k(x)[y]$ are the nonzero members of $k(x)$.

We claim that f and g have no irreducible factor in common in $k(x)[y]$. Suppose for contradiction that h is such a factor, so that in particular $h \notin k(x)$. Since all nonzero elements of $k[x]$ are units in $k(x)[y]$ we may as well suppose that $h \in k[x, y]$ (multiply through by something suitable). Now h can be factorised into irreducibles in the UFD $k[x, y]$ and at least one factor is not in $k[x]$, so we may as well assume that h is irreducible in $k[x, y]$.

So suppose that $ha = f$ and $hb = g$ for some $a, b \in k(x)[y]$. Multiply by a suitable nonzero P in $k[x]$ to get $hA = fP$, $hB = fP$ where $A, B \in k[x, y]$. Now $h \notin k[x]$ so h cannot divide P, thus h divides both f and g. Contradiction.

Now since $k(x)[y]$ is a PID 1 is in the ideal generated by f and g in $k(x)[y]$, say $1 = cf + dg$ for $c, d \in k(x)[y]$. Multiplying by a suitable nonzero $Q \in k[x]$ we get $Q = Cf + Dg$ for $C, D \in k[x, y]$.

Finally any point at which both f and g vanish has an x-coordinate where Q vanishes. So there are only finitely many possible x-coordinates, and similarly for y.

3: (Thanks to Richard D for the nice example) Let $R = \mathbb{Z}[x_1, x_2, \ldots]$ (the classsic example of a non-Noetherian ring). Let $T = R[y]$ and let $S = R[yx_1, yx_2, \ldots]$. Clearly $R \subseteq S \subseteq T$ and T is ring-finite over R. We claim that S is not ring-finite over R. The point is that if A is a finite subset of S then A can only mention x_i for i less than some large N; it is easy to see that yx_N is not in $R[A]$.