From a Mesoscopic to a Macroscopic Description of Fluid-Particle Interaction

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Outline

Assumptions on Fluid and Particles

- Fluid is inviscid and compressible
- Only one type of particle in the fluid
- Particles are uniform spheres with density ϱ_P and radius a
- Fixed spatial domain $\Omega \subseteq \mathbb{R}^3$
- System is at a fixed, constant temperature $\theta_0 > 0$.
- Fluid is described by density $\varrho(x, t) \in [0, \infty)$ and velocity field $\mathbf{u}(x, t) \in \mathbb{R}^3$.

Description of Particles

- Particle distribution in the fluid is described by density function

$$f(x, \xi, t) \in [0, \infty)$$

  where ξ is the microscopic velocity fluctuation.
- Particles subject to Brownian motion, leading to diffusion in ξ, with diffusion constant

$$\frac{k\theta_0}{m_p}\frac{6\pi\mu a}{m_p} = \frac{k\theta_0}{m_p}\frac{9\mu}{2a^2\varrho_P}$$

  where k is the Boltzmann constant and μ the dynamic viscosity of the fluid.
- Macroscopic particle density $\eta(x, t)$ given by

$$\eta(x, t) := \int_{\mathbb{R}^3} f(x, \xi, t) \, \mathrm{d}\xi. \quad (1)$$

Coupling of Fluid and Particles

Coupling of the system is due to the friction between the particles and the fluid following Stokes' Law

Definition (Stokes' Law)

Consider a uniform, spherical particle of radius a. The friction force exerted on a particle by the fluid is

$$F(x, \xi, t) = 6\pi\mu a[\mathbf{u}(x, t) - \xi] \tag{2}$$

Thus, the force exerted on the fluid by the particles is

$$6\pi\mu a \int_{\mathbb{R}^3} [\xi - \mathbf{u}(x, t)] f(x, \xi, t) \; \mathrm{d}\xi$$

by Newton's Third Law.

External Force–Physical Assumptions

Both fluid and particles are influenced by an external force with a time independent potential $\Phi(x)$.

- Force exerted on a particle: $-m_P \nabla_x \Phi$.
- Force exerted per unit volume on fluid: $\alpha \varrho_F \nabla_x \Phi$.
  - α: dimensionless constant measuring ratio of external force's strength on fluid and particles
  - ϱ_F: typical value of fluid mass per volume
- Measures settling phenomena such as gravity, buoyancy, and centrifugal forces.

Vlassov-Euler System

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{3}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varrho_F} \nabla_x p(\varrho) = -\alpha \varrho \nabla_x \Phi + \frac{6\pi\mu a}{\varrho_F} \int_{\mathbb{R}^3} (\xi - \mathbf{u}) f \, \mathrm{d}\xi \tag{4}$$

$$\partial_t f + \xi \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_\xi f = \frac{9\mu}{2a^2 \varrho_P} \mathrm{div}_\xi \left[(\xi - \mathbf{u}) f + \frac{k\theta_0}{m_P} \nabla_\xi f \right] \tag{5}$$

We assume a pressure of the form $p(\varrho) = \kappa \varrho^\gamma$ where $\kappa > 0$ and $\gamma > 1$.

Unitless Parameters I

In order to find a macroscopic model, we transform (3)-(5) to a unitless model with dimensionless parameters, and scale these parameters, then take the appropriate limit.

- Stokes settling time

$$\mathcal{T}_S = \frac{m_P}{6\pi\mu a} = \frac{2\varrho_P a^2}{9\mu}$$

- Thermal speed

$$\mathcal{V}_{th} = \sqrt{\frac{k\theta_0}{m_P}}$$

Unitless Parameters II

- We also define characteristic time T, length L, and velocity $U = L/T$.
- We define a pressure unit $\mathcal{P}$ and associate a velocity $\mathcal{V}_S$ to the external potential Φ.

Thus, the physical values in relation to the dimensionless parameters are (' indicates unitless quantity)

$$
\begin{aligned}
&t = Tt' && x = Lx' \\
&\xi = \mathcal{V}_{th}\xi' && \varrho(Lx', Tt') = \varrho'(x', t') \\
&\mathbf{u}(Lx', Tt') = U\mathbf{u}'(x', t) && p(Lx', Tt') = \mathcal{P}p'(x', t') \\
&f(x', \xi', t') = \tfrac{4}{3}\pi a^3 \mathcal{V}_{th}^3 f(Lx', \mathcal{V}_{th}\xi', Tt') && \Phi(Lx') = \tfrac{\mathcal{V}_S L}{T_S}\Phi'(x')
\end{aligned}
$$

Unitless Parameters III

We also define the unitless constants

$$\beta = \frac{T}{L}\mathcal{V}_{th} = \frac{\mathcal{V}_{th}}{U}, \quad \frac{1}{\varepsilon} = \frac{T}{\mathcal{T}_S}, \quad n = \frac{\mathcal{V}_S T}{\mathcal{V}_{th}\mathcal{T}_S}, \quad \chi = \frac{\mathcal{P}T}{\varrho_F L U} = \frac{\mathcal{P}}{\varrho_F U^2}$$

Unitless Vlassov-Euler System

Using the previous relations in (3)-(5) yields after dropping primes

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{6}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\chi p(\varrho)) = -\alpha\beta n \varrho \nabla_x \Phi + \frac{1}{\varepsilon}\frac{\varrho_P}{\varrho_F}\int_{\mathbb{R}^3}(\beta\xi - \mathbf{u}) f \, \mathrm{d}\xi \tag{7}$$

$$\partial_t f + \beta\xi \cdot \nabla_x f - n \nabla_x \Phi \cdot \nabla_\xi f = \frac{1}{\varepsilon}\mathrm{div}_\xi\left[\left(\xi - \frac{1}{\beta}\mathbf{u}\right) f + \nabla_\xi f\right] \tag{8}$$

Pressure and Internal Energy

- The enthalpy is defined as

$$h(\varrho) := \int_1^{\varrho} \frac{p'(s)}{s} \, \mathrm{d}s$$

and is in $L^1_{\mathrm{loc}}(0,\infty)$.

- The internal energy is defined as

$$\Pi(\varrho) := \int_0^{\varrho} h(s) \, \mathrm{d}s$$

Free Energy

Assume that

$$\frac{\varrho_P}{\varrho_F} = \frac{1}{\beta^2} \text{ and } n = \beta.$$

The free energies are

- The fluid free energy
$$\mathcal{F}_F(\varrho, \mathbf{u}) = \int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \chi\Pi(\varrho) + \alpha\beta^2\varrho\Phi \,\mathrm{d}x \tag{9}$$
- The particle free energy
$$\mathcal{F}_P(f) = \int_\Omega \int_{\mathbb{R}^3} f \ln f + \frac{|\xi|^2}{2} f + f\Phi \,\mathrm{d}\xi \,\mathrm{d}x \tag{10}$$
- The total free energy
$$\mathcal{F}(\varrho, \mathbf{u}, f) = \mathcal{F}_F(\varrho, \mathbf{u}) + \mathcal{F}_P(f) \tag{11}$$

Energy Dissipation

Theorem

Assuming the scaling above, we have the following dissipation.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F} + \frac{1}{\varepsilon}\int_{\Omega}\int_{\mathbb{R}^3}\left|(\xi - \beta^{-1}\mathbf{u})\sqrt{f} + 2\nabla_{\xi}\sqrt{f}\right|^2 \,\mathrm{d}\xi\,\mathrm{d}x \leq 0. \quad (12)$$

This result follows formally from integration by parts.

External Force–Mathematical Assumptions

- $\exp(-\Phi) \in L^1(\Omega)$, $\Phi \exp(-\Phi) \in L^1(\Omega)$.
- $\Phi \in W^{1,1}(\Omega)$ for bounded Ω; $\Phi \in W^{1,1}_{\mathrm{loc}}(\Omega)$ for unbounded Ω.
- $\alpha\Omega$ is bounded below on Ω.
- The sub-level sets of $\alpha\Phi$ are bounded, that is

$$\{x \in \Omega \mid \alpha\Phi \le k\}$$

is bounded for any $k \in \mathbb{R}$.

Stationary Solutions I

Provided the total fluid mass is conserved in time and finite, the system (6)-(8) has a stationary solution $(\varrho_s, \mathbf{u}_s, f_s)$ such that

- $\mathbf{u}(x) \equiv 0$
- The stationary particle density function is

$$f_s(x,\xi) = Z_P e^{-\Phi(x)} \frac{e^{-|\xi|^2/2}}{(2\pi)^{3/2}}$$

where

$$Z_P = \left(\int_\Omega \int_{\mathbb{R}^3} f_0 \, \mathrm{d}\xi \, \mathrm{d}x \right) \left(\int_\Omega e^{-\Phi(x)} \, \mathrm{d}x \right)^{-1}.$$

Stationary Solutions II

- The stationary fluid density is given by

$$\varrho_s(x) = \sigma\left(Z_F - \frac{\alpha\beta n}{\chi}\Phi(x)\right)$$

where

$$Z_F = \left(\int_\Omega \varrho_0 \,\mathrm{d}x\right)\left(\int_\Omega e^{-\Phi(x)} \,\mathrm{d}x\right)^{-1}$$

and σ is the generalized inverse of h.

Estimates I

Assuming that

$$\int_{\Omega}\int_{\mathbb{R}^3} f_0 \left(1 + |\ln(f_0)| + \frac{|\xi|^2}{2} + |\Phi|\right)\, \mathrm{d}\xi\, \mathrm{d}x$$

and

$$\int_{\Omega} \varrho_0 + \varrho_0|\mathbf{u}_0|^2 + |\Pi(\varrho_0)| + \varrho_0\beta n|\alpha\Phi|\, \mathrm{d}x$$

are both bounded, we can use the free energy inequality and the hypotheses on Φ to obtain the following uniform bounds

Estimates II

- $f(1+|\xi|^2+|\Phi|+|\ln f|)$ is bounded in $L^\infty(\mathbb{R}^+;L^1(\Omega\times\mathbb{R}^3))$
- ϱ, $|\Pi(\varrho)|$ and $\beta n\varrho|\alpha\Phi|$ are bounded in $L^\infty(\mathbb{R}^+;L^1(\Omega))$
- $\sqrt{\varrho}\mathbf{u}$ is bounded in $L^\infty(\mathbb{R}^+;L^2(\Omega))$
- $\frac{1}{\sqrt{\varepsilon}}\left[\left(\xi-\beta^{-1}\mathbf{u}\right)\sqrt{f}+2\nabla_\xi\sqrt{f}\right]$ is bounded in $L^2(\mathbb{R}^+\times\Omega\times\mathbb{R}^3)$.

Expansions of Macroscopic Particle Quantities

We define the unitless first moment of f as

$$\mathbf{J}(x,t) = \beta \int_{\mathbb{R}^3} \xi f(x,\xi,t) \, \mathrm{d}\xi$$

and unitless second moment

$$\mathbb{P}(x,t) = \int_{\mathbb{R}^3} \xi \otimes \xi \; f(x,\xi,t) \, \mathrm{d}\xi.$$

Using the uniform bounds, these quantities can be expanded as

$$\mathbf{J} = \mathbf{u}\eta + \beta\sqrt{\varepsilon}\mathbf{K}$$

and

$$\mathbb{P} = \eta\mathbb{I} + \beta^{-2}\mathbf{J} \otimes \mathbf{u} + \sqrt{\varepsilon}\mathbb{K}$$

where the components of $\mathbf{K}$ and $\mathbb{K}$ are bounded in $L^2(\mathbb{R}^+; L^1(\Omega))$.

Flowing Regime Scaling

- We are interested when the settling time scale is much smaller than the observational time scale, that is

  $$\mathcal{T}_S \ll T$$

  so ε is small.
- We are interested then in the limit $\varepsilon \to 0$.
- We take $\beta^2 = \varrho_F/\varrho_P$ to be a constant and $n = \beta$ a fixed positive constant.
- Thus, $\mathcal{V}_S \ll U = \mathcal{V}_{th}$.
- ϱ_F and ϱ_P are of the same order.

Flowing Regime Vlassov-Euler System

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{13}$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x(\chi p(\varrho_\varepsilon)) = -\alpha\beta^2 \varrho_\varepsilon \nabla_x \Phi + \frac{1}{\varepsilon\beta^2}\int_{\mathbb{R}^3} (\xi - \mathbf{u}_\varepsilon) f_\varepsilon \, \mathrm{d}\xi \tag{14}$$

$$\partial_t f_\varepsilon + \beta(\xi \cdot \nabla_x f_\varepsilon - \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon) = \frac{1}{\varepsilon} \mathrm{div}_\xi \left[\left(\xi - \frac{1}{\beta}\mathbf{u}_\varepsilon \right) f_\varepsilon + \nabla_\xi f_\varepsilon \right] \tag{15}$$

Macroscopic Limit of Flowing Regime

Assuming that the limits of the various unknown quantities and their non-linear combinations involved in the system exist, the limits ϱ, $\mathbf{u}$, and η as $\varepsilon \to 0$ are

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{16}$$

$$\partial_t[(\varrho + \beta^{-2}\eta)\mathbf{u}] + \mathrm{div}_x[(\varrho + \beta^{-2}\eta)\mathbf{u} \otimes \mathbf{u}] + \nabla_x(\chi p(\varrho) + \eta) = -(\alpha\beta^2\varrho + \eta)\nabla_x \Phi \tag{17}$$

$$\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u}) = 0 \tag{18}$$

Bubbling Regime Scaling

- Again, the settling time scale is much smaller than the observational time scale, that is

$$\mathcal{T}_S \ll T$$

so ε is small.

- We are interested then in the limit $\varepsilon \to 0$.
- Again, $\beta^2 = \varrho_F / \varrho_P$ and $n = \beta$, but $\beta = \varepsilon^{-1/2}$ and $\alpha = \text{sgn}(\alpha)\varepsilon$.
- Physically, $\mathcal{V}_S = U \ll \mathcal{V}_{th}$.

Bubbling Regime Vlassov-Euler System

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{19}$$

$$\begin{aligned}\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x(\chi p(\varrho_\varepsilon)) \\ = -\,\mathrm{sgn}(\alpha)\varrho_\varepsilon \nabla_x \Phi + \int_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - \mathbf{u}_\varepsilon \right) f_\varepsilon \, \mathrm{d}\xi\end{aligned} \tag{20}$$

$$\begin{aligned}\partial_t f_\varepsilon + \frac{1}{\sqrt{\varepsilon}}(\xi \cdot \nabla_x f_\varepsilon + \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon) \\ = \frac{1}{\varepsilon} \mathrm{div}_\xi [(\xi - \sqrt{\varepsilon}\mathbf{u}_\varepsilon) f_\varepsilon + \nabla_\xi f_\varepsilon]\end{aligned} \tag{21}$$

Macroscopic Limit of Bubbling Regime

Assuming that the limits pass as $\varepsilon \to 0$,

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{22}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\chi p(\varrho) + \eta) = -(\mathrm{sgn}(\alpha)\varrho + \eta)\nabla_x \Phi \tag{23}$$

$$\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u} - \eta \nabla_x \Phi) = \Delta_x \eta \tag{24}$$

Remarks

- We assume that f_ε, ϱ_ε, and ε converge to f, ϱ, and $\mathbf{u}$ in each regime, as well as any non-linear terms converge. While we have weak compactness for the sequences (f_ε), (ϱ_ε), $(\mathbf{u}_\varepsilon)$, and $(\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon)$ from the energy inequality, this is not the case for the non-linear terms.
- Such rigor can be shown in the case of a viscous fluid.
- Even in the case of no external force, the evolution of the fluid still depends on the evolution of the particle density.

References

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