Local and Global Existence of Solutions for the Compressible Euler-Smoluchowski Model

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Outline

Fluid-Particle Interaction

- Fluid-particle interaction models are of interest to engineers and scientists studying biotechnolgy, medicine, waste-water recycling, mineral processing, and combustion theory.
- The macroscopic model considered in this talk, the Euler-Smoluchowski system, is formally derived from a Fokker-Planck type kinetic equation coupled with fluid equations.
- This coupling is from the mutual frictional forces between the particles and the fluid, assumed to follow Stokes' Law.
- The fluid is an inviscid, compressible fluid.

Motivation: Euler System with Friction

Consider the system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) + a\varrho \mathbf{u} = 0$$

- This system can be symmetrized, and by the results of Majda, has a smooth solution for finite time.
- Sideris *et al*. show that the friction force $-a\varrho \mathbf{u}$ $(a > 0)$ can be used to obtain a global existence result.

Euler-Smoluchowski Model

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(a\varrho^\gamma + \eta) = -(\beta\varrho + \eta)\nabla_x \Phi \tag{2}$$

$$\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u} - \eta \nabla_x \Phi) = \Delta_x \eta \tag{3}$$

This model is derived in the paper of Carillo and Goudon.
We consider the Cauchy problem on $\mathbb{R}^3$ and assume that the initial data $\varrho_0, \mathbf{u}_0, \eta_0$ have support on some compact set $K \subset \mathbb{R}^3$.

Additional Difficulties with NSS System

- The NSS system cannot be written as a hyperbolic system since the Smoluchowski equation (3) is parabolic. Thus, the results of Friedrichs, Kato, and Majda cannot be used directly.
- The external forcing term is more complicated than a friction force. Thus, the dissipation assumption on the force is more delicate and needs more physical justification.

Symmetrizing the ES System I

If we write the ES system in matrix form as

$$\mathbb{A}_0(U)\partial_t U + \sum_{i=1}^{3} \mathbb{A}_i(U)\partial_{x_i} U + \mathbb{A}_4 = 0 \tag{4}$$

where, using $\mathbb{A}_1$ as an example,

$$\mathbb{A}_1 := \begin{bmatrix} \mathbf{u}_1 & \varrho & 0 & 0 & 0 \\ p_F'(\varrho) & \varrho\mathbf{u}_1 & 0 & 0 & 1 \\ 0 & 0 & \varrho\mathbf{u}_1 & 0 & 0 \\ 0 & 0 & 0 & \varrho\mathbf{u}_1 & 0 \\ 0 & \eta & 0 & 0 & \mathbf{u}_1 - \partial_{x_1}\Phi \end{bmatrix},$$

we see that this system is not symmetric.

Symmetrizing the ES System II

Using the standard transformation for Euler systems

$$w := \frac{2}{\gamma - 1}\left(\sqrt{p_F'(\varrho)} - \overline{\sigma}\right) \tag{5}$$

leaves us with a system in matrix form with

$$\mathbb{B}_1 := \begin{bmatrix} \mathbf{u}_1 & \overline{\sigma} + \frac{\gamma-1}{2}w & 0 & 0 & 0 \\ \overline{\sigma} + \frac{\gamma-1}{2}w & \mathbf{u}_1 & 0 & 0 & f(w) \\ 0 & 0 & \mathbf{u}_1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{u}_1 & 0 \\ 0 & \eta & 0 & 0 & \mathbf{u}_1 - \partial_{x_1}\Phi \end{bmatrix}.$$

Symmetrizing the ES System III

However, if we consider the ES with η being given, we get the matrix form such that

$$\mathbb{B}_1 := \begin{bmatrix} \mathbf{u}_1 & \overline{\sigma} + \frac{\gamma-1}{2} w & 0 & 0 \\ \overline{\sigma} + \frac{\gamma-1}{2} w & \mathbf{u}_1 & 0 & 0 \\ 0 & 0 & \mathbf{u}_1 & 0 \\ 0 & 0 & 0 & \mathbf{u}_1 \end{bmatrix}.$$

Motivation for Approximation Scheme

To seed the iterative construction of approximate local solutions, we set $\mathbf{u}^0$ as the solution to

$$\begin{aligned} \partial_t \mathbf{v} - \Delta_x \mathbf{v} &= 0 \\ \mathbf{v}(x,0) &= \mathbf{u}_0. \end{aligned} \tag{6}$$

We then find η^1 using (9), and put this in to (7)-(8) to find $w^1, \mathbf{u}^1$. We use $\mathbf{u}^1$ to find η^2 and so on.

Approximate ES System

Consider the system

$$\partial_t w^k + \overline{\sigma}\,\mathrm{div}_x\,\mathbf{u}^k = -\mathbf{u}^k \cdot \nabla_x w^k - \frac{\gamma - 1}{2} w^k\,\mathrm{div}_x\,\mathbf{u}^k \qquad (7)$$

$$\partial_t \mathbf{u}^k + \overline{\sigma}\nabla_x w^k + [\beta + f(w^k)\eta^k]\nabla_x \Phi = -(\mathbf{u}^k \cdot \nabla_x)\mathbf{u}^k - \frac{\gamma - 1}{2} w^k \nabla_x w^k \qquad (8)$$

$$\partial_t \eta^k + \eta^k\,\mathrm{div}_x(\mathbf{u}^{k-1} - \nabla_x \Phi) + (\mathbf{u}^{k-1} - \nabla_x \Phi) \cdot \nabla_x \eta^k - \Delta_x \eta^k = 0. \qquad (9)$$

Local Existence of Approximate Solutions I

If we know η^k and if w_0 and $\mathbf{u}_0$ have high enough regularity, we have the following theorem due to the work of Friedrichs, Kato, and Majda.

Theorem (Solutions for Symmetric Hyperbolic Systems)

Let $w_0 \in W^{3,2}(\Omega)$ and $\mathbf{u}_0 \in W^{3,2}(\Omega; \mathbb{R}^3)$ with the support of w_0 and $\mathbf{u}_0$ contained in some compact subset K of Ω. Assume also that $\eta^k \in C^1([0,T]; C^2(\Omega))$. Then there is a time interval $[0,T]$ with $T > 0$ such that there is a unique classical solution

$$\begin{aligned} w^k &\in C([0,T]; W^{3,2}(\Omega)) \cap C^1([0,T]; W^{2,2}(\Omega)) \\ \mathbf{u}^k &\in C([0,T]; W^{3,2}(\Omega; \mathbb{R}^3)) \cap C^1([0,T]; W^{2,2}(\Omega; \mathbb{R}^3)). \end{aligned} \tag{10}$$

Further, T depends only on w_0, $\mathbf{u}_0$ and K.

Local Existence of Approximate Solutions II

Theorem (Existence of Approximate Smooth Solutions)

Let

$$w_0 \in W^{3,2}(\Omega)$$
$$\mathbf{u}_0 \in W^{3,2}(\Omega; \mathbb{R}^3)$$
$$\eta_0 \in W^{3,2}(\Omega)$$

all with support contained in some compact subset K of Ω. Let $\mathbf{u}^0 \in C^1([0,T]; C^2(\Omega))$ solve (6). Then there exists some $T > 0$ such that for all $k \in \mathbb{N}$, there exist solutions $\{w^k, \mathbf{u}^k, \eta^k\}$ of (7)-(9) such that

$$w^k \in C([0,T]; W^{3,2}(\Omega)) \cap C^1([0,T]; W^{2,2}(\Omega))$$
$$\mathbf{u}^k \in C([0,T]; W^{3,2}(\Omega; \mathbb{R}^3)) \cap C^1([0,T]; W^{2,2}(\Omega; \mathbb{R}^3))$$
$$\eta^k \in C^1([0,T]; C^2(\Omega)).$$

Further, T depends only on w_0, $\mathbf{u}_0$ and K.

Convergence of Approximate Solutions

To obtain local solutions to the ES system, we take $k \to \infty$. We obtain estimates on the quantities

$$\overline{w}^k := w^{k+1} - w^k$$
$$\overline{\mathbf{u}}^k := \mathbf{u}^{k+1} - \mathbf{u}^k$$
$$\overline{\eta}^k := \eta^{k+1} - \eta^k$$

We can then obtain convergence as $k \to \infty$ and obtain the following theorem.

Existence of Local Solutions

Theorem

Let w_0*,* $\eta_0 \in W^{3,2}(\Omega)$ *and* $\mathbf{u}_0 \in W^{3,2}(\Omega; \mathbb{R}^3)$ *all with support in some compact subset* K *of* Ω*. Then there is some* $T > 0$ *such that there exists a solution* $\{w, \mathbf{u}, \eta\}$ *to the symmetrized ES system such that*

$$w \in C([0, T]; W^{3,2}(\Omega)) \cap C^1([0, T]; W^{2,2}(\Omega))$$

$$\mathbf{u} \in C([0, T]; W^{3,2}(\Omega; \mathbb{R}^3)) \cap C^1([0, T]; W^{2,2}(\Omega; \mathbb{R}^3))$$

$$\eta \in C([0, T]; W^{3,2}(\Omega)) \cap C^1([0, T]; W^{2,2}(\Omega)).$$

Global In-Time Existence of Smooth Solutions

Weak-Dissipation Hypothesis

- Assume that the external force satisfies the relation

$$\mathbf{u} \cdot \nabla_x \Phi \geq 0$$

- Physically, this can be seen as the external force acting against the fluid velocity (c.f. friction or drag)
- Like the paper of Sideris and Thomases, which considers the external force of friction, this condition on the external force allows for the existence of global smooth solutions.
- This result depends upon showing a finite propagation speed, achieved with a Gronwall's argument.

References

1) J. Ballew and K. Trivisa, Viscous and inviscid models in fluid-particle interaction. Submitted to *Communications in Information Systems*, 2013.

2) J. A. Carrillo and T. Goudon. Stability and Asymptotic Analysis of a Fluid-Particle Interaction Model. *Comm. Partial Differential Equations*, **31**:1349–1379, 2006.

3) A. Majda. *Compressible fluid flow and systems of conservation laws in several space variables.* Applied Mathematical Sciences, Vol. 53, Springer-Verlag, New York, 1984.

4) T. C. Sideris, B. Thomases, and D. Wang. Long time behavior of solutions to the 3D compressible Euler equations with Damping. *Comm. Partial Diff. Eqs.* **Vol. 28, Nos. 3 and 4**:795-816, 2003.

Inequality for Finite Propagation Speed

$$
\begin{aligned}
&\frac{1}{2}\int_{|y-x|\le\overline{\sigma}(t-\tau)} w^2+|\mathbf{u}|^2+\eta^2\,\mathrm{d}y-\frac{1}{2}\int_{|y-x|\le\overline{\sigma}t} w^2+|\mathbf{u}|^2+\eta^2\,\mathrm{d}y\\
&+\overline{\sigma}\int_0^\tau\int_{|y-x|=\overline{\sigma}(t-s)}\frac{1}{2}(w^2+|\mathbf{u}|^2+\eta^2)+w\mathbf{u}\cdot\frac{y-x}{|y-x|}\,\mathrm{d}S_y\,\mathrm{d}s\\
&-\frac{\overline{\sigma}}{\sqrt{\overline{\sigma}^2+1}}\int_0^\tau\int_{|y-x|=\overline{\sigma}(t-s)}(\eta^2\nabla_x\Phi+\eta\nabla_x\eta)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}S_y\,\mathrm{d}s\\
&\le\int_0^\tau\int_{|y-x|\le\overline{\sigma}(t-s)}\frac{1}{2}\eta\nabla_x\Phi\cdot\nabla_x\eta-\eta^2\,\mathrm{div}_x\,\mathbf{u}-\beta\mathbf{u}\cdot\nabla_x\Phi\,\mathrm{d}y\,\mathrm{d}s\\
&-\int_0^\tau\int_{|y-x|\le\overline{\sigma}(t-s)}\eta f(w)\mathbf{u}\cdot\nabla_x\Phi+f(w)\mathbf{u}\cdot\nabla_x\eta+f(w)\eta\mathbf{u}\cdot\nabla_x\Phi\,\mathrm{d}y\,\mathrm{d}s\\
&-\int_0^\tau\int_{|y-x|\le\overline{\sigma}(t-s)}\frac{\gamma-1}{2}w\mathbf{u}\cdot\nabla_x w+\frac{\gamma-1}{2}w^2\,\mathrm{div}_x\,\mathbf{u}\,\mathrm{d}y\,\mathrm{d}s\\
&-\int_0^\tau\int_{|y-x|\le\overline{\sigma}(t-s)}\mathbf{u}\cdot(\mathbf{u}\cdot\nabla_x\mathbf{u})+\eta\mathbf{u}\cdot\nabla_x\eta\,\mathrm{d}y\,\mathrm{d}s. \qquad (11)
\end{aligned}
$$