LOW MACH NUMBER LIMITS TO THE NAVIER-STOKES-SMOLUCHOWSKI SYSTEM

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Abstract. This article presents a general dimensionless scaling of the Navier-Stokes-Smoluchowski system describing interactions between particles and a compressible fluid. Two low Mach number limits are investigated. The first limit is a low stratification limit for which the Froude number is scaled as the square root of the Mach number; the second is a strong stratification limit for which the Froude and Mach numbers are scaled the same. We see that as the Mach number goes to zero in the low stratification case, the solutions to the system converge in appropriate spaces to constant mass densities and weakly to a velocity field satisfying the incompressibility condition. For the strong stratification case, we see for an external force depending only on the vertical coordinate that the solutions converge to densities depending only on the vertical component and a velocity field satisfying the anelastic condition. Finally, we investigate bounds and convergences for the strong stratification case supporting the formal calculations.

1. **Introduction.** The state of fluid-particle-interaction flows is characterized by the following macroscopic variables: the total mass density $\varrho(t,x)$, the velocity field $\mathbf{u}(t,x)$, and the density of particles dispersed in the mixture $\eta(t,x)$, which depend on the Eulerian spatial coordinate $x \in \Omega \subset \mathbb{R}^3$ and on time $t \in (0,\infty)$. The governing equations express the conservation of mass, the balance of momentum, and the balance of particle densities often referred to as the *Smoluchowski equation*:

$$\partial_t \varrho + \operatorname{div}_x (\varrho \mathbf{u}) = 0 \tag{1.1}$$

$$\partial_t (\varrho \mathbf{u}) + \operatorname{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \left(p_F(\varrho) + \frac{D}{\zeta}\eta \right) = \mu \triangle_x \mathbf{u} + \lambda \nabla_x \operatorname{div}_x \mathbf{u} - (\eta + \beta\varrho) \nabla_x \Phi \tag{1.2}$$

$$\partial_t \eta + \operatorname{div}_x (\eta (\mathbf{u} - \zeta \nabla_x \Phi)) - D \triangle_x \eta = 0 \tag{1.3}$$

where $p_F(\varrho) = a\varrho^\gamma$ for some $a > 0$, $\gamma > \frac{3}{2}$, and $\beta \neq 0$. We also assume a bounded $C^{2,\nu}$ spatial domain Ω. The fluid is also assumed to be Newtonian so that the stress tensor is given by

$$\mathbb{S} = \mu(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^T) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I}.$$

Also, the viscosity coefficients μ and λ, the drag coefficient ζ, and the dispersion coefficient D are assumed to be constant, and Φ is a given external potential that is

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taken to be nonnegative. The system (1.1)-(1.3) is supplemented by the following boundary and initial conditions:

$$\mathbf{u} = D\nabla_x \eta \cdot \mathbf{n} + \zeta\eta\nabla_x \Phi \cdot \mathbf{n} = 0 \text{ on } (0,T)\times\partial\Omega \tag{1.4}$$

$$0 \leq \varrho(0,x) = \varrho_0 \in L^\gamma(\Omega) \tag{1.5}$$

$$(\varrho\mathbf{u})(0,x) = \mathbf{m}_0 \in L^{6/5}(\Omega;\mathbb{R}^3) \tag{1.6}$$

$$0 \leq \eta(0,x) = \eta_0 \in L^2(\Omega). \tag{1.7}$$

We define the energy

$$\mathcal{E}(t) := \int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \frac{D}{\zeta}\eta\ln\eta + (\beta\varrho+\eta)\Phi \mathrm{d}x(t) \tag{1.8}$$

and require that

$$\frac{\mathrm{d}E}{\mathrm{d}t} + \int_\Omega \mu|\nabla_x\mathbf{u}|^2 + \lambda|\mathrm{div}_x\mathbf{u}|^2 + \left|\frac{2D}{\sqrt{\zeta}}\nabla_x\sqrt{\eta} + \sqrt{\eta}\nabla_x\Phi\right|^2 \mathrm{d}x \leq 0.$$

In addition, we require that the spatial domain Ω and external potential Φ obey the following hypotheses, called the *confinement hypotheses*:

Definition 1.1. Let $\Omega \subset \mathbb{R}^3$ be a $C^{2,\nu}$ domain with $\nu > 0$ and $\Phi : \Omega \to \mathbb{R}_0^+$ with $\inf_{x\in\Omega}\Phi(x) = 0$. (Ω,Φ) satisfies the **Confinement Hypotheses (HC)** if and only if

- If Ω is bounded, Φ is bounded and Lipschitz continuous on $\overline{\Omega}$.
- If Ω is unbounded, $\Phi \in W^{1,\infty}_{\text{loc}}(\Omega)$, $e^{-\Phi/2} \in L^1(\Omega)$ and

$$|\Delta_x\Phi(x)| \leq c_1|\nabla_x\Phi(x)| \leq c_2\Phi(x)$$

for $|x|$ greater than some large R.

In [4] it is shown using an artificial pressure and time-discretization approximation that a *renormalized weak solution* exists. In [3], a weak-strong uniqueness result is shown on the NSS system; that is, if there is a weak solution of a certain regularity class, the the weak solution is unique.

The rest of the paper is dedicated to examining certain approximations to the compressible NSS system in the form of singular limits for bounded spatial domains Ω. In particular, we look at conditions for which the speed of the fluid flow is small compared to the speed of sound in the fluid, also known as the low Mach number case. Under a low stratification condition of the scaling of the system, the solutions converge to a solution of the mathematically simpler incompressible fluid model as the Mach number approaches zero. In the strong stratification case, the solutions will converge to functions obeying the anelastic condition, if we assume that the external force depends only on the vertical component of position, physically realized for buoyancy and gravity near the surface of the earth or other similar body. Both of these problems involve using bounds from the energy inequality for the systems to provide estimates that allow us to show the convergence of the solutions. These techniques are motivated by the work in [5, 6, 7, 8].

2. **Dimensionless Scaling.** For each parameter α (time, length, mass, density, pressure, etc.), we define a reference value α_{ref} and then define the dimesionless value

$$\alpha' := \frac{\alpha}{\alpha_{\text{ref}}}.$$

By using the chain rule and basic differentiation properties, the NSS system in terms of the dimensionless parameters and values becomes (with the prime marks omitted)

$$\mathrm{Sr}\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{2.9}$$

$$\mathrm{Sr}\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x\left(a\varrho^\gamma + \mathrm{Pc}\frac{D}{\zeta}\eta\right) = \frac{1}{\mathrm{Re}}(\mu\Delta_x\mathbf{u} + \lambda\nabla_x\mathrm{div}_x\mathbf{u}) - \frac{1}{\mathrm{Fr}^2}(\beta\varrho + \mathrm{Dc}\eta)\nabla_x\Phi \tag{2.10}$$

$$\mathrm{Sr}\partial_t\eta + \mathrm{div}_x(\eta\mathbf{u}) - \mathrm{Za}\mathrm{div}_x(\zeta\eta\nabla_x\Phi) - \mathrm{Da}D\Delta_x\eta = 0 \tag{2.11}$$

with the scaled energy inequality

$$\mathrm{Sr}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \mathrm{Pc}\frac{D\eta}{\zeta}\ln\eta + \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}(\beta\varrho + \mathrm{Dc}\eta)\Phi\mathrm{d}x$$
$$+ \int_\Omega \mathrm{PcDa}D^2\frac{|\nabla_x\eta|^2}{\zeta\eta} + 2\mathrm{Za}D\nabla_x(\eta)\cdot\nabla_x\Phi + \frac{\mathrm{Za}^2}{\mathrm{Da}}\zeta\eta|\nabla_x\Phi|^2\mathrm{d}x$$
$$+ \int_\Omega \frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\mathrm{d}x \le 0$$

with the unitless coefficients defined in the following table.

Sr:= $\frac{L_{ref}}{\mathbf{u}_{ref}t_{ref}}$	Ma:= $\frac{\mathbf{u}_{ref}}{\sqrt{p_{F_{ref}}/\varrho_{ref}}}$	Re:= $\frac{\varrho_{ref}\mathbf{u}_{ref}L_{ref}}{\mu_{ref}}$
Fr:= $\frac{\mathbf{u}_{ref}}{\sqrt{L_{ref}f_{ref}}}$	Za:= $\frac{\zeta_{ref}f_{ref}}{\mathbf{u}_{ref}}$	Da:= $\frac{D_{ref}}{L_{ref}\mathbf{u}_{ref}}$
Pc:= $\frac{p_{P_{ref}}}{p_{F_{ref}}}$	Dc:= $\frac{\eta_{ref}}{\varrho_{ref}}$.	

Table 2.1: Definitions of the Dimensionless Parameters

3. **Low Stratification Limit.** The scaled low stratification system we consider for each fixed $\varepsilon > 0$ is

$$\partial_t\varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon) = 0 \tag{3.12}$$

$$\varepsilon^2[\partial_t(\varrho_\varepsilon\mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon)] + \nabla_x\left(a\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon\right) = \varepsilon^2(\mu\Delta_x\mathbf{u}_\varepsilon + \lambda\nabla_x\mathrm{div}_x\mathbf{u}_\varepsilon) - \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x\Phi \tag{3.13}$$

$$\partial_t\eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon\mathbf{u}_\varepsilon) - \varepsilon\mathrm{div}_x(\zeta\eta_\varepsilon\nabla_x\Phi) - D\Delta_x\eta_\varepsilon = 0 \tag{3.14}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta}\ln\eta_\varepsilon + \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi\mathrm{d}x$$
$$+ \int_\Omega D^2\frac{|\nabla_x\eta_\varepsilon|^2}{\zeta\eta_\varepsilon} + 2\varepsilon D\nabla_x\eta_\varepsilon\cdot\nabla_x\Phi + \varepsilon^2\zeta\eta_\varepsilon|\nabla_x\Phi|^2\mathrm{d}x$$
$$+ \int_\Omega \varepsilon^2\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) : \nabla_x\mathbf{u}_\varepsilon\mathrm{d}x \le 0. \tag{3.15}$$

To rigorously derive the limit for the low stratification case, we begin by noting that from the results of [4], for each $\varepsilon > 0$, we have solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ in the following sense:

Definition 3.1. We say that $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ is a *renormalized weak solution to the scaled low stratification NSS system* if and only if

- $\varrho_\varepsilon \geq 0$ and $\mathbf{u}_\varepsilon$ form a renormalized solution of the scaled continuity equation, i.e.,
$$\int_0^T \int_\Omega B(\varrho_\varepsilon)\partial_t\varphi + B(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi - b(\varrho_\varepsilon)\mathrm{div}_x\mathbf{u}_\varepsilon\varphi \mathrm{d}x\mathrm{d}t$$
$$= -\int_\Omega B(\varrho_0)\varphi(0,\cdot)\mathrm{d}x \tag{3.16}$$
where $b \in L^\infty \cap C[0,\infty)$, $B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z$.
- The scaled momentum balance holds in the sense of distribution.
- $\eta_\varepsilon \geq 0$ is a weak solution of the scaled Smoluchowski equation.
- The scaled energy inequality (3.15) is satisfied.

We next define the *low stratification target system.*

Definition 3.2. $\{\varrho^{(1)}, \overline{\mathbf{u}}, \eta^{(1)}\}$ solve the *low stratification target system* if and only if

$$\mathrm{div}_x\overline{\mathbf{u}} = 0 \text{ weakly on } (0,T)\times\Omega,$$
$$\int_0^T \int_\Omega \overline{\varrho}\overline{\mathbf{u}} \cdot \partial_t\mathbf{v} + \overline{\varrho}\overline{\mathbf{u}} \otimes \overline{\mathbf{u}} : \nabla_x\mathbf{v}\mathrm{d}x\mathrm{d}t$$
$$= \int_0^T \int_\Omega (\mu\nabla_x\overline{\mathbf{u}} - (\beta r + s)\nabla_x\Phi)\cdot\mathbf{v}\mathrm{d}x\mathrm{d}t - \int_\Omega \overline{\varrho}\overline{\mathbf{u}}\cdot\mathbf{v}(0,\cdot)\mathrm{d}x,$$

for any divergence-free text function $\mathbf{v}$ and

$$r = -\frac{1}{a\gamma\overline{\varrho}^{\gamma-1}}\left[(\beta\overline{\varrho} + \overline{\eta})\Phi + \frac{D}{\zeta}s\right]$$

weakly where $\overline{\varrho}$ and $\overline{\eta}$ are uniform fluid and particle densities, respectively, with the same total masses as the initial data.

We are now in a position to state the main theorem of this section.

Theorem 3.3. *Let (Ω,Φ) satisfy the confinement hypothesis and for each $\varepsilon > 0$, assume $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \eta_\varepsilon\}$ is a solution of the low stratification system in the sense of Definition 3.1. Assume the initial data can be expressed as follows:*

$$\varrho_\varepsilon(0,\cdot) = \varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon\varrho^{(1)}_{\varepsilon,0},\ \boldsymbol{u}_\varepsilon(0,\cdot) = \boldsymbol{u}_{\varepsilon,0},\ \textit{and}\ \eta_\varepsilon(0,\cdot) = \eta_{\varepsilon,0} = \overline{\eta} + \varepsilon\eta^{(1)}_{\varepsilon,0}.$$

where $\overline{\varrho}, \overline{\eta}$ are the spatially uniform densities on Ω. Assume also that as $\varepsilon \to 0$,

$$\varrho^{(1)}_{\varepsilon,0} \rightharpoonup \varrho^{(1)}_0,\ \boldsymbol{u}_{\varepsilon,0} \rightharpoonup \overline{\boldsymbol{u}}_0, \eta^{(1)}_{\varepsilon,0} \rightharpoonup \eta^{(1)}_0$$

weakly-$*$ in $L^\infty(\Omega)$ or $L^\infty(\Omega;\mathbb{R}^3)$ as the case may be. Then up to a subsequence and letting $q := \min\{\gamma, 2\}$,

$$\varrho_\varepsilon \to \overline{\varrho} \textit{ in } C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^q(\Omega))$$
$$\eta_\varepsilon \to \overline{\eta} \textit{ in } L^2(0,T;L^2(\Omega))$$
$$\boldsymbol{u}_\varepsilon \to \overline{\boldsymbol{u}} \textit{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$$

and

$$\varrho^{(1)}_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \textit{weakly-}*\ \textit{ in } L^\infty(0,T;L^q(\Omega))$$
$$\eta^{(1)}_\varepsilon = \frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon} \to \eta^{(1)} \textit{ weakly in } L^2(0,T;L^2(\Omega))$$

where $\{\varrho^{(1)}, \overline{\boldsymbol{u}}, \eta^{(1)}\}$ solve the target system mentioned previously.

Proof. For the proof, the reader may consult [2]. □

4. **Strong Stratification Limit.** The formal calculations for the strong stratification limit as $\varepsilon \to 0$ follow the same procedure as for the low stratification limit. We use the following scaling: Ma is taken to be a small parameter $\varepsilon > 0$, Za and Da are taken to be ε^{-1}, Fr is taken to be ε, and other parameters are taken to be of order 1. We also assume that $\Phi = gx_3$ where g is a constant (gravity/buoyancy). Thus, the scaled NSS system becomes

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{4.17}$$

$$\varepsilon^2[\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] + \nabla_x \left(a\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon \right)$$

$$= \varepsilon^2(\mu \Delta_x \mathbf{u}_\varepsilon + \lambda \nabla_x \mathrm{div}_x \mathbf{u}_\varepsilon) - (\beta \varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \tag{4.18}$$

$$\varepsilon\left[\partial_t \eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon)\right] - \mathrm{div}_x(\zeta \eta_\varepsilon \nabla_x \Phi) - D\Delta_x \eta_\varepsilon = 0 \tag{4.19}$$

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma - 1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta} \ln \eta_\varepsilon + (\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi \mathrm{d}x$$

$$+ \varepsilon \int_\Omega \varepsilon^2 \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \mathrm{d}x + \int_\Omega \left| D\frac{\nabla_x \eta_\varepsilon}{\sqrt{\zeta \eta_\varepsilon}} + \sqrt{\zeta \eta_\varepsilon} \nabla_x \Phi \right|^2 \mathrm{d}x \le 0. \tag{4.20}$$

Now, assuming $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ have the following expansions

$$\varrho_\varepsilon = \tilde{\varrho} + \sum_{i=1}^\infty \varepsilon^i \varrho_\varepsilon^{(i)}$$

$$\eta_\varepsilon = \tilde{\eta} + \sum_{i=1}^\infty \varepsilon^i \eta_\varepsilon^{(i)}$$

$$\mathbf{u}_\varepsilon = \tilde{\mathbf{u}} + \sum_{i=1}^\infty \varepsilon^i \mathbf{u}_\varepsilon^{(i)}$$

we substitue into (4.17)-(4.20) and formally obtain the target system

$$g\tilde{\eta} = -\frac{D}{\zeta}\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}x_3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[a\tilde{\varrho}^\gamma\right] = -\beta g \tilde{\varrho}$$

$$\mathrm{div}_x(\tilde{\varrho}\tilde{\mathbf{u}}) = 0$$

$$\tilde{\varrho}\partial_t \tilde{\mathbf{u}} + \mathrm{div}_x(\tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \nabla_x \Pi = \mu \Delta_x \tilde{\mathbf{u}} + \lambda \nabla_x \mathrm{div}_x \tilde{\mathbf{u}} - \left(\beta \varrho^{(2)} + \eta^{(2)}\right) \nabla_x \Phi.$$

For the strong stratification scaling, we have the following weak formulation:

Definition 4.1. We say that $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ form a *renormalized weak solution to the scaled strong stratification NSS system* if and only if

- $\varrho_\varepsilon \ge 0$ and $\mathbf{u}_\varepsilon$ form a renormalized solution of the scaled continuity equation, i.e.,

$$\int_0^T \int_\Omega B(\varrho_\varepsilon)\partial_t \varphi + B(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \varphi - b(\varrho_\varepsilon)\mathrm{div}_x \mathbf{u}_\varepsilon \varphi \mathrm{d}x\mathrm{d}t$$

$$= -\int_\Omega B(\varrho_0)\varphi(0,\cdot)\mathrm{d}x \tag{4.21}$$

  where $b \in L^\infty \cap C[0,\infty)$, $B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z$.

- The scaled momentum balance holds in the sense of distributions.
- $\eta_\varepsilon \geq 0$ is a weak solution of the scaled Smoluchowski equation.
- The scaled energy inequality (4.20) is satisfied.

Note that for this scaling, we assume that $\Phi = gx_3$, where x_3 is the vertical coordinate, and g is a constant greater than zero. We also define the target system.

Definition 4.2. $\{\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta}, \varrho^{(2)}, \eta^{(2)}\}$ solve the *strong stratification target system* if and only if:

- $$\int_0^T \int_\Omega \tilde{\varrho}\tilde{\mathbf{u}} \cdot \nabla_x \phi \mathrm{d}x\mathrm{d}t = 0 \tag{4.22}$$
  for all $\phi \in C_C^\infty((0,T) \times \Omega)$,
- $$g\tilde{\eta} = -\frac{D}{\zeta}\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}x_3} \tag{4.23}$$
  $$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[a\tilde{\varrho}^\gamma\right] = -\beta g \tilde{\varrho} \tag{4.24}$$
  with the conditions
  $$\int_\Omega \tilde{\varrho}\mathrm{d}x = \int_\Omega \varrho_0 \mathrm{d}x$$
  $$\int_\Omega \tilde{\eta}\mathrm{d}x = \int_\Omega \eta_0 \mathrm{d}x,$$
- $$\begin{aligned}&\int_0^T \int_\Omega \tilde{\varrho}\tilde{\mathbf{u}} \cdot \mathbf{w} + \tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \nabla_x \mathbf{w} \mathrm{d}x\mathrm{d}t \\ &= \int_0^T \int_\Omega \mu \nabla_x \tilde{\mathbf{u}} \nabla_x \mathbf{w} - \left(\beta \varrho^{(2)} + \eta^{(2)}\right) \nabla_x \Phi \cdot \mathbf{w} \mathrm{d}x\mathrm{d}t\end{aligned} \tag{4.25}$$
  for all $\mathbf{w} \in C_C^\infty((0,T) \times \Omega; \mathbb{R}^3)$ such that $\mathrm{div}_x \mathbf{w} = 0$.

Much like for the low stratification limit, many of the bounds and convergences used in the analysis arise from the free energies defined as

$$E_F(\varrho, \tilde{\varrho}) := \frac{a}{\gamma - 1}\varrho^\gamma - (\varrho - \tilde{\varrho})\frac{a\gamma}{\gamma - 1}\tilde{\varrho}^{\gamma-1} - \frac{a}{\gamma - 1}\tilde{\varrho}^\gamma$$

$$E_P(\eta, \tilde{\eta}) := \frac{D}{\zeta}\eta \ln \eta - \frac{D}{\zeta}(\eta - \tilde{\eta})(\ln \tilde{\eta} + 1) - \frac{D}{\zeta}\tilde{\eta}\ln\tilde{\eta},$$

and the resulting inequality formed from these and the energy inequality:

$$\begin{aligned}&\int_\Omega \frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left[E_F(\varrho_\varepsilon, \tilde{\varrho}) + E_P(\eta_\varepsilon, \tilde{\eta})\right] \mathrm{d}x(T) \\ &\int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \mathrm{d}x\mathrm{d}t + \frac{1}{\varepsilon^3}\int_0^T \int_\Omega \left|\frac{D\nabla_x \eta_\varepsilon}{\sqrt{\zeta\eta_\varepsilon}} + \sqrt{\zeta\eta_\varepsilon}\nabla_x \Phi\right|^2 \mathrm{d}x\mathrm{d}t \\ &\leq \int_\Omega \frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}[E_F(\varrho_0, \tilde{\varrho}) + E_P(\eta_0, \tilde{\eta})]\mathrm{d}x.\end{aligned} \tag{4.26}$$

Next, we define the essential and residual sets:

$$\mathcal{O}_{\text{ess}} := \{(\varrho,\eta) \in \mathbb{R}^2 | \tilde{\varrho}/2 \le \varrho \le 2\tilde{\varrho}, \tilde{\eta}/2 \le \eta \le 2\tilde{\eta}\}$$
$$\mathcal{M}^{\varepsilon}_{\text{ess}} := \{(x,t) \in (0,T) \times \Omega | (\varrho_\varepsilon(t,x), \eta_\varepsilon(t,x)) \in \mathcal{O}_{\text{ess}}\}$$
$$\mathcal{M}^{\varepsilon}_{\text{res}} := ((0,T) \times \Omega) - \mathcal{M}^{\varepsilon}_{\text{ess}}$$

Thus, by using (4.26), assuming appropriate bounds on the initial data, we obtain that

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$$
$$\|[\varrho_\varepsilon - \tilde{\varrho}]\text{ess}\|_{L^\infty(0,T;L^2(\Omega))} \le \varepsilon^2 c$$
$$\|[\eta_\varepsilon - \tilde{\eta}]\text{ess}\|_{L^\infty(0,T;L^2(\Omega))} \le \varepsilon^2 c$$
$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))$$
$$\left\| \frac{D\nabla_x \eta_\varepsilon}{\sqrt{\zeta\eta_\varepsilon}} + \sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi \right\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le \varepsilon^3 c$$
$$\left\{ \left[\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon}\right]_{\text{ess}} \right\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega))$$
$$\left\{ \left[\frac{\eta_\varepsilon - \tilde{\eta}}{\varepsilon}\right]_{\text{ess}} \right\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega))$$

and since the measure of the residual set goes as ε^2 for each fixed t, we have

$$\|[\varrho_\varepsilon]\text{res}\|_{L^\infty(0,T;L^\gamma(\Omega))} \le \varepsilon^2 c$$
$$\{\varrho_\varepsilon \mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^\infty(0,T;L^{2q/q+1}(\Omega;\mathbb{R}^3)) \cap L^{6q/q+6}(\Omega;\mathbb{R}^3))$$

where $q := \min\{2,q\}$. Thus, we have the existence of $\varrho^{(1)}, \eta^{(1)} \in L^\infty(0,T;L^2(\Omega))$ and $\tilde{\mathbf{u}} \in L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))$ such that up to subsequences

$$\varrho_\varepsilon \to \tilde{\varrho} \text{ strongly in } L^\infty(0,T;L^q(\Omega))$$
$$\eta_\varepsilon \to \tilde{\eta} \text{ strongly in } L^\infty(0,T;L^2(\Omega))$$
$$\mathbf{u}_\varepsilon \rightharpoonup \tilde{\mathbf{u}} \text{ weakly in } L^2(0,T;W^{1,2}_0(\Omega;\mathbb{R}^3))$$
$$\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon} \rightharpoonup \varrho^{(1)} \text{ weakly-}* \text{ in } L^\infty(0,T;L^q(\Omega))$$
$$\frac{\eta_\varepsilon - \tilde{\eta}}{\varepsilon} \rightharpoonup \varrho^{(1)} \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Now, we are in a position to state the main result of this section:

Theorem 4.3. *Let (Ω,Φ) satisfy the confinement hypothesis and for each $\varepsilon > 0$, assume $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \eta_\varepsilon\}$ solves the scaled strong stratification system in the sense of Definition 4.1. Assume the initial data can be expressed as follows:*

$$\varrho_\varepsilon(0,\cdot) = \varrho_{\varepsilon,0} = \tilde{\varrho} + \varepsilon\varrho^{(1)}_{\varepsilon,0},\ \boldsymbol{u}_\varepsilon(0,\cdot) = \boldsymbol{u}_{\varepsilon,0},\ \textit{and}\ \eta_\varepsilon(0,\cdot) = \eta_{\varepsilon,0} = \tilde{\eta} + \varepsilon\eta^{(1)}_{\varepsilon,0}.$$

where $\tilde{\varrho}, \tilde{\eta}$ are the densities defined by (4.24)-(4.23). Assume also that as $\varepsilon \to 0$,

$$\varrho^{(1)}_{\varepsilon,0} \rightharpoonup \varrho^{(1)}_0, \boldsymbol{u}_{\varepsilon,0} \rightharpoonup \tilde{\boldsymbol{u}}_0, \eta^{(1)}_{\varepsilon,0} \rightharpoonup \eta^{(1)}_0$$

weakly-$*$ in $L^\infty(\Omega)$ or $L^\infty(\Omega;\mathbb{R}^3)$ as the case may be. Then up to a subsequence and letting $q := \min\{\gamma, 2\}$,

$$\varrho_\varepsilon \to \tilde{\varrho} \textit{ in } C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^q(\Omega))$$

$$\eta_\varepsilon \to \tilde{\eta} \text{ in } L^2(0,T;L^2(\Omega))$$
$$\boldsymbol{u}_\varepsilon \to \tilde{\boldsymbol{u}} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$$

where $\{\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta}\}$ *solve the target system* (4.22)-(4.25).

Proof. The result follows from the bounds listed above and analysis similar to that done in Section 3 and in [8]. For the details, see [1] □

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