VANISHING VISCOSITY SOLUTIONS FOR THE NAVIER-STOKES-SMOLUCHOWSKI SYSTEM FOR PARTICLES IN A COMPRESSIBLE FLUID

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Abstract. The existence of measure-valued solutions obeying an entropy balance for the one-dimensional Euler-Smoluchowski system for particles dispersed in a compressible fluid is shown. These measure-valued solutions are derived from a Young measure arising from solutions to the corresponding Navier-Stokes-Smoluchowski system for particles dispersed in a viscous, compressible fluid which are shown to exist by using bounds uniform in the viscosity coefficient. The Young measure is then shown to reduce to a delta mass for adiabatic constants greater than one.

1. Introduction

Fluid-particle interaction phenomena arise is several areas of science, including sedimentation analysis, biotechnology, medicine, waste-water recycling, mineral processing, atmospheric sciences, and combustion of fuel droplets [2, 3, 8, 9, 23, 24]. The friction forces the particles and fluid exert mutually on each other lead to a coupling to the fluid and kinetic equations. The models considered in this paper assume that the friction force follows Stokes' Law and is proportional to the relative velocity, that is, the fluctuations of the microscopic velocity. The cloud of particles is described by a distribution f which is the solution to a Vlasov-Fokker-Planck equation in the viscous case. The particle density η is the integral over the microscopic velocity of f. For more detail, the interested reader is referred to [6, 10, 21].

The one-dimensional Euler-Smoluchowski system for compressible, inviscid fluids on the spatial domain $\mathbb{R}$ is

$$\partial_t \varrho + \partial_x(\varrho u) = 0 \tag{1.1a}$$

$$\partial_t(\varrho u) + \partial_x \left(\varrho u^2 + a\varrho^\gamma + \eta\right) = -(\beta\varrho + \eta)\Phi_x \tag{1.1b}$$

$$\partial_t \eta + \partial_x (\eta u - \eta \Phi_x) = \partial_{xx}\eta. \tag{1.1c}$$

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The corresponding Navier-Stokes-Smoluchowski system for compressible, viscous fluids is

$$\partial_t \varrho + \partial_x(\varrho u) = 0 \tag{1.2a}$$

$$\partial_t(\varrho u) + \partial_x(\varrho u^2 + a\varrho^\gamma + \eta) = \varepsilon \partial_{xx} u - (\beta\varrho + \eta)\Phi_x \tag{1.2b}$$

$$\partial_t \eta + \partial_x(\eta u - \eta \Phi_x) = \partial_{xx}\eta. \tag{1.2c}$$

In (1.1)-(1.2), the quantities ϱ, u, and η are the unknown fluid density, fluid velocity, and macroscopic particle density, respectively, while $\gamma > 1$ is the adiabatic constant and a is a constant greater than zero, which in this paper is taken to be $\frac{(\gamma-1)^2}{4\gamma}$ by choosing appropriate physical units. The external potential Φ is a non-negative function on $\mathbb{R}$ subject to the following confinement hypothesis.

Definition 1.1 (Confinement Hypothesis)**.** The external potential $\Phi : \mathbb{R} \mapsto \mathbb{R}^+$ and the constant β are said to satisfy the confinement hypothesis if and only if

- $\beta > 0$,
- Φ is locally Lipschitz continuous,
- Φ is bounded above,
- $e^{-\Phi/2} \in L^1(\mathbb{R})$, and
- Φ_x is compactly supported.

Remark 1.1. It is noted that the conditions in Definition 1.1 are stronger than the conditions in [7, 10], particularly the compact support of Φ_x. This is necessary because of the imposition of a positive background fluid density.

As will be seen in Section 3, Definition 1.1 ensures that a lower bound exists on the $\eta \ln \eta$ term in the energy, which enables the use of an energy functional for obtaining uniform estimates for solutions to (1.2).

The solutions of (1.2) depend on the viscosity coefficient ε, but for notational simplicity, there will be no ε subscripts on solutions to (1.2) when there is no chance of confusion. It is assumed that $\varepsilon \in (0, \varepsilon_0]$ for some $\varepsilon_0 > 0$.

For (1.1) and (1.2), the boundary conditions at $x = \pm\infty$ are

$$u(x,t) \to 0 \tag{1.3a}$$

$$\eta(x,t)\Phi(x) + \partial_x \eta(x,t) \to 0 \tag{1.3b}$$

$$\varrho(x,t) \to \overline{\varrho}, \tag{1.3c}$$

where $\overline{\varrho}$ is a positive constant are imposed. Further, it is assumed that

$$\int_{\mathbb{R}} \eta_0 \, dx = M_p < \infty \tag{1.4a}$$

$$\int_{\mathbb{R}} \varrho_0 - \overline{\varrho} \, dx = M_f < \infty. \tag{1.4b}$$

Consequences of (1.3) and (1.4) include conservation of particle mass (see [7, 10]) and the conservation of the relative fluid mass

$$\int_{\mathbb{R}} \varrho - \overline{\varrho}\ dx.$$

Existence and uniqueness results for (1.2) and (1.1) have been shown in previous work. In [10], existence of weak solutions obeying an energy inequality to the three-dimensional verison of (1.2) are shown along with asymptotic long-time convergence of these solutions to a steady-state solution. Existence of weakly dissipative solutions obeying relative entropy inequality to the three-dimensional version of (1.2) along with a weak-strong uniqueness result is shown in [7]. In [6], existence of local-in-time smooth solutions to (1.1) in three space dimensions is shown for appropriately regular initial data.

In the current paper, the issue of vanishing viscosity limits for solutions to (1.2) is investigated. This investigation follows those for the Navier-Stokes systems in [12, 13]. The main result of the current paper is as follows.

Theorem 1.1. *Let* $(\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon)$ *be a sequence of smooth solutions to* (1.2) *with initial data* (1.4) *and conditions at* $x = \pm\infty$ (1.3) *on* $\mathbb{R} \times (0,T)$ *for* $T > 0$ *for* $\varepsilon \in (0, \varepsilon_0]$ *for some* $\varepsilon_0 > 0$*. Assume further that* $\sqrt{\varepsilon}\frac{\eta_\varepsilon}{\varrho_\varepsilon}$ *is bounded in* $L^2((0,T)\times\mathbb{R})$ *uniformly in* ε *and* $\frac{\eta_\varepsilon}{\varrho_\varepsilon}$ *is uniformly bounded in* ε *on* $\mathbb{R} \times (0,T)$*. Assume also the following conditions on the initial data.*

(1) *There is a positive constant* $E_0 < \infty$ *such that the initial relative mechanical energy satisfies*

$$\int_{\mathbb{R}} \frac{1}{2}\varrho_0|u_0|^2 + e^*(\varrho_0, \overline{\varrho}) + \eta_0 \ln \eta_0 + \eta_0 \Phi(x)\ dx \le E_0 \tag{1.5}$$

where $e^*(\varrho, \overline{\varrho}) \stackrel{def}{=} \frac{a}{\gamma-1}\varrho^\gamma - \frac{a}{\gamma-1}\overline{\varrho}^\gamma - \frac{a\gamma}{\gamma-1}\overline{\varrho}^{\gamma-1}\varrho + \frac{a\gamma}{\gamma-1}\overline{\varrho}^\gamma$ *is the relative specific internal energy of the fluid.*

(2) *There is a positive constant* $E_1 < \infty$ *such that*

$$\varepsilon^2 \int_{\mathbb{R}} \frac{|\partial_x \varrho_0|^2}{\varrho_0^3}\ dx \le E_1. \tag{1.6}$$

(3) *There is a positive constant* $E_2 < \infty$ *such that*

$$\varepsilon \int_{\mathbb{R}} \frac{\eta_0^2}{\varrho_0^2}\ dx \le E_2. \tag{1.7}$$

Then there is a subsequence (not relabeled) $(\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon)$ *that converges almost everywhere to an entropy solution* (ϱ, u, η) *to* (1.1) *obeying* (1.3) *and* (1.4).

Remark 1.2. The bounds on $\frac{\eta_\varepsilon}{\varrho_\varepsilon}$, while not explicitly shown to hold, are related to the scaling of the physical constants in the bubbling regime, which is under consideration in this paper. The interested reader is referred to [11] for more details.

Theorem 1.1 is analogous to the main results in [12] and [13] for the one-dimensional Navier-Stokes and Euler systems on the real line and the corresponding three-dimensional problem with radial symmetry, respectively. However, while Chen and Perepelitsa in [12, 13] are able to easily use the variety of entropy/entropy-flux pairs for the Euler system of two unknowns, the system (1.1) under consideration here has three unknowns, limiting the entropy/entropy-flux pair to that from the mechanical energy. Thus, definition of the measure-valued solutions is modified from that used in [12, 13], as will be seen. The structure of the rest of the paper, which is dedicated to the proof of Theorem 1.1 is as follows.

(1) In Section 2, background on entropy/entropy-flux pairs is developed. In addition, results for entropy/entropy-flux pairs based on an entropy kernel for a system of two equations in two unknowns are stated.
(2) Section 3 is dedicated to the proofs of several estimates on $(\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon)$ which are independent of ε. These estimates are crucial in allowing passage to the limits in the following sections.
(3) Sections 4 and 5 develop the compensated compactness framework necessary for the existence of the measure-valued solutions.
(4) In Section 6, the convergence of solutions to (1.2) to measure-valued solutions is shown. In addition, the commutator relation key to showing the reduction of the Young measure to a delta mass is calculated.
(5) Lastly, in Section 7, the Young measures constructed are shown to reduce to delta masses, allowing for the completion of the proof of Theorem 1.1.

2. Entropies and Entropy/Entropy Flux Pairs

This section collects some well-known results for entropy/entropy-flux pairs (see [14], for example) and some bounds on a family of entropy/entropy-flux pairs that will be used for the Euler-Smoluchowski system (1.1). For this section, if the the continuity and momentum equations (1.1a)-(1.1b) are considered with the variable η as a fixed function of space and time, the resulting system

$$\partial_t U + \partial_x F(U) = \begin{bmatrix} 0 \\ -\partial_x \eta - (\beta\varrho + \eta)\Phi_x \end{bmatrix} \tag{2.1}$$

where $U = [\varrho, m = \varrho u]^T$ and $F(U) = \left[m, \frac{m^2}{\varrho} + p(\varrho)\right]^T$ is hyperbolic. It can be shown (see [12, Section 2] and [14]) that the eigenvalues for this system are

$$\lambda_\pm = u \pm \theta\varrho^\theta \tag{2.2}$$

and the Riemann invariants are

$$w_\pm = u \mp \varrho^\theta \tag{2.3}$$

where $\theta \stackrel{\text{def}}{=} \frac{\gamma-1}{2}$. Since $\lambda_+ - \lambda_- = 2\theta\varrho^\theta$, the system is strictly hyperbolic for $\varrho > 0$ and loses the its strict hyperbolicity at the vacuum state $\varrho = 0$.

A pair of functions (H, Q) is called an entropy-entropy flux pair for (2.1) if

$$DQ(U) = (DF(U))^T\, DH(U) \tag{2.4}$$

where D is the total differentiation operator in the components of U and

$$DF \stackrel{\text{def}}{=} \begin{bmatrix} \partial_\varrho F_1 & \partial_m F_1 \\ \partial_\varrho F_2 & \partial_m F_2 \end{bmatrix}$$

is the Jacobian of F in the coordinates (ϱ, m).

One example of an entropy/entropy-flux pair is the mechanical energy H^* and mechanical energy flux Q^* in the variables ϱ and u given by

$$H^*(\varrho, m) = \frac{1}{2}\frac{m^2}{\varrho} + \frac{a}{\gamma-1}\varrho^\gamma$$

$$Q^*(\varrho, m) = \frac{1}{2}\frac{m^3}{\varrho^2} + \frac{a\gamma}{\gamma-1}\varrho^\gamma u.$$

It is also noted that

$$\mathcal{E}(\varrho, u, \eta) \stackrel{\text{def}}{=} \frac{1}{2}\varrho u^2 + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \ln \eta + (\beta\varrho + \eta)\Phi$$

is an entropy to (1.1) (see [5, 6, 7], for example).

Considering an entropy H as a function of ϱ and u, H will obey the integrability condition (see [14, Chapter 16.8])

$$\partial_{\varrho\varrho} H = \theta^2 \varrho^{\gamma-3} \partial_{uu} H. \tag{2.6}$$

It is clear that (2.6) is singular across the axis $\varrho = 0$, and the nature of the singularity depends upon γ. Also of interest in light of (2.6) are weak entropies of (2.1) that vanish for $\varrho = 0$. These entropies are given by

$$H^\psi(\varrho, \varrho u) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \chi(\varrho; s - u)\psi(s)\, ds \tag{2.7a}$$

$$Q^\psi(\varrho, \varrho u) \stackrel{\text{def}}{=} \int_{\mathbb{R}} (\theta s + (1-\theta)u)\chi(\varrho; s - u)\psi(s)\, ds \tag{2.7b}$$

for any continuous function ψ, where χ is the weak entropy kernel determined by

$$\partial_{\varrho\varrho}\chi - \frac{p'(\varrho)}{\varrho^2}\partial_{uu}\chi = 0 \tag{2.8a}$$

$$\chi(0, u; s) = 0 \tag{2.8b}$$

$$\partial_\varrho \chi(0, u; s) = \delta_{u=s} \tag{2.8c}$$

where $\delta_{u=s}$ is the delta mass concentrated at $u = s$. Thus, for the γ-law case under consideration in this paper, the weak entropy kernel is given by (see [12, Section 2] and [14, Chapter 16.8])

$$\chi(\varrho; s - u) = \left[\varrho^{2\theta} - (s - u)^2\right]_+^\lambda \tag{2.9}$$

where $\lambda \stackrel{\text{def}}{=} \frac{3-\gamma}{2(\gamma-1)}$, so clearly $\lambda > -\frac{1}{2}$. Thus, for $\psi \in C(\mathbb{R})$,

$$H^\psi(\varrho, \varrho u) = \int_{\mathbb{R}} \left[\varrho^{2\theta} - (s-u)^2\right]_+^\lambda \psi(s)\, ds = \varrho \int_{-1}^{1} \psi(u + s\varrho^\theta)(1 - s^2)^\lambda\, ds \tag{2.10}$$

and

$$\begin{aligned} Q^\psi(\varrho, \varrho u) &= \int_{\mathbb{R}} (\theta s + (1-\theta)u)\left[\varrho^{2\theta} - (s-u)^2\right]_+^\lambda \psi(s)\, ds \\ &= \varrho \int_{-1}^{1} (u + \theta\varrho^\theta s)\psi(u + s\varrho^\theta)(1 - s^2)^\lambda\, ds. \end{aligned} \tag{2.11}$$

For $\psi \in C_c(\mathbb{R})$, the following estimates hold (see [12, Lemma 2.1]).

Lemma 2.1. *Let $\psi \in C_c(\mathbb{R})$ such that* $\operatorname{supp}\psi \subset [a, b]$. *Then the supports of H^ψ and Q^ψ are contained in the set*

$$\left\{(\varrho, m) = (\varrho, \varrho u) : \varrho^\theta + u \geq a,\ u - \varrho^\theta \leq b\right\}.$$

In addition, there exists a constant $C_\psi > 0$ depending only on ψ such that for any $(\varrho, u) \in [0, \infty) \times \mathbb{R}$,

- *if* $\gamma \in (1,3]$,

$$\left|H^{\psi}(\varrho,m)\right| + \left|Q^{\psi}(\varrho,m)\right| \le C_{\psi}; \tag{2.12}$$

- *if* $\gamma > 3$,

$$\left|H^{\psi}(\varrho,m)\right| \le C_{\psi}\varrho \tag{2.13}$$

and

$$\left|Q^{\psi}(\varrho,m)\right| \le C_{\psi}\max\{1,\varrho^{\theta}\}; \tag{2.14}$$

- *if* $\psi \in C_c^2(\mathbb{R})$, *then*

$$\left|\partial_m H^{\psi}(\varrho,m)\right| + \left|\varrho\partial_{mm} H^{\psi}(\varrho,m)\right| \le C_{\psi}; \tag{2.15}$$

- *if* $\psi \in C_c^2(\mathbb{R})$, *then*

$$\left|\partial_{mu} H^{\psi}(\varrho,\varrho u)\right| + \left|\varrho^{1-\theta}\partial_{m\varrho}(\varrho,\varrho u)\right| \le C_{\psi}. \tag{2.16}$$

The proof uses the same calculations using (2.10) and (2.11) as in the proof of Lemma 2.1 in [12] and is omitted here.

Taking $\psi(s) = \psi_{\#}(s) \stackrel{\text{def}}{=} \frac{1}{2}s|s|$, the following lemma (from [12] and [20]) gives estimates on $H^{\#} \stackrel{\text{def}}{=} H^{\psi_{\#}}$ and $Q^{\#} \stackrel{\text{def}}{=} Q^{\psi_{\#}}$.

Lemma 2.2. *For* $\psi_{\#}$, (2.10) *and* (2.11) *give the following estimates:*

$$|H^{\#}(\varrho,\varrho u)| \le C(\varrho u^2 + \varrho^{\gamma}) \tag{2.17a}$$

$$|\partial_m H^{\#}(\varrho,\varrho u)| \le C(|u| + \varrho^{\theta}) \tag{2.17b}$$

$$|\partial_{mm} H^{\#}(\varrho,\varrho u)| \le C\varrho^{-1} \tag{2.17c}$$

and considering $\partial_m H^{\#}$ *as a function of* ϱ *and* u,

$$|\partial_{um} H^{\#}(\varrho,\varrho u)| \le C \tag{2.18a}$$

$$|\partial_{\varrho m} H^{\#}(\varrho,\varrho u)| \le C\varrho^{\theta-1} \tag{2.18b}$$

where C *is a positive constant depending only on* γ.

The proof involves straight-forward estimates using (2.10) and is omitted here.

Finally, using the work of Lions, Perthame, and Tadmor [20, Lemma 4], it can be shown that there is some constant C dependent only upon γ such that

$$Q^{\#}(\varrho,u) \ge C(\varrho^{3\theta+1} + \varrho^{\gamma}|u| + \varrho^{\theta+1}|u|^2 + \varrho|u|^3). \tag{2.19}$$

3. Uniform Estimates

This section of uniform estimates begins with an energy estimate that allows for various terms to be estimated independent of ε. Throughout this section, the subscript ε on the solutions to (1.2) is omitted. Before these estimates are proven, a result following from the confinement hypothesis is needed. This lemma controls the negative part of the $\eta\ln\eta$ term, which controls the other quantities in the energy estimate to follow.

Lemma 3.1. *Let Φ satisfy the conditions in Definition 1.1 and let η be any non-negative function in $L^1(\mathbb{R})$. Then*

$$\int_{\mathbb{R}} \eta \ln_- \eta \; dx \le \frac{1}{2}\int_{\mathbb{R}} \eta\Phi \; dx + \frac{1}{e}\int_{\mathbb{R}} e^{-\Phi/2} \; dx \tag{3.1}$$

where $\ln_- \eta$ is the negative part of $\ln\eta$.

This lemma is Lemma 3.6 in [10] and is proven there and in [15]. As such, the proof is omitted here.

Proposition 3.1. *For smooth solutions ϱ, u, and η of (1.2) on $\mathbb{R}\times[0,T]$ for some $T > 0$, there exists some positive constant C independent of ε such that*

$$\sup_{t\in[0,T]} E[\varrho,u,\eta](t) + \int_0^T \int_{\mathbb{R}} \varepsilon|\partial_x u|^2 \; dx \; dt + \int_0^T \int_{\mathbb{R}} \left|\frac{\partial_x\eta}{\sqrt{\eta}} + \sqrt{\eta}\Phi_x\right|^2 dx \; dt \le C \tag{3.2}$$

where

$$\begin{aligned} &E[\varrho,u,\eta](t) \\ &\quad \stackrel{def}{=} \int_{\mathbb{R}} \frac{1}{2}\varrho(x,t)|u(x,t)|^2 + e^*(\varrho,\overline{\varrho}) + \eta(x,t)\ln\eta(x,t) + \eta(x,t)\Phi(x) \; dx. \end{aligned} \tag{3.3}$$

Proof. Let ϱ, u, and η be solutions to (1.2) for some fixed ε. Noting that the integrand in (3.3) is just the relative entropy $\mathcal{H}$ with respect to the end states $\overline{\varrho}$, 0:

$$\mathcal{H}(\varrho,u|\overline{\varrho},0) \stackrel{\text{def}}{=} H^*(\varrho,u) - H^*(\overline{\varrho},0) - DH^*(\overline{\varrho},0)\cdot\begin{bmatrix}\varrho-\overline{\varrho}\\ \varrho u\end{bmatrix}, \tag{3.4}$$

it is clear that

$$\begin{aligned} &\frac{d}{dt}E(\varrho,u,\eta) = \\ &\quad \int_{\mathbb{R}} \partial_t(H^*(\varrho,u) + \eta\ln\eta + \eta\Phi) - \partial_t H^*(\overline{\varrho},0) - DH^*(\overline{\varrho},0)\cdot\begin{bmatrix}\partial_t\varrho\\ \partial_t(\varrho u)\end{bmatrix} \; dx. \end{aligned} \tag{3.5}$$

Clearly, the second term in the integral of the right side of (3.5) is zero, and noting that

$$-\int_{\mathbb{R}} DH^*(\overline{\varrho},0)\cdot\begin{bmatrix}\partial_t\varrho\\ \partial_t(\varrho u)\end{bmatrix} \; dx = \int_{\mathbb{R}} \varrho u\partial_x DH^*(\overline{\varrho},0) \; dx = 0, \tag{3.6}$$

then

$$\frac{d}{dt}E(\varrho,u,\eta) = \frac{d}{dt}\int_{\mathbb{R}} \frac{1}{2}\varrho|u|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \eta\ln\eta + \eta\Phi \; dx. \tag{3.7}$$

Multiplying (1.2b) by u, noting that

$$\partial_x(a\varrho^\gamma)u = \partial_t\left(\frac{a}{\gamma-1}\varrho^\gamma\right) + \partial_x\left(\frac{a\gamma}{\gamma-1}\varrho^\gamma u\right) \tag{3.8}$$

by using (1.2a), and deriving from (1.2c) that

$$\begin{aligned} &(\partial_x\eta + \eta\Phi_x)\,u \\ &\quad = \partial_t(\eta\ln\eta+\eta\Phi) + \partial_x[(\ln\eta+1+\Phi)(\eta u - \eta\Phi_x - \partial_x\eta)] + \left(\frac{\partial_x\eta}{\sqrt{\eta}} + \sqrt{\eta}\Phi_x\right)^2, \end{aligned} \tag{3.9}$$

the entropy equation

$$\begin{aligned}(3.10)\quad &\partial_t\left(\frac{1}{2}\varrho|u|^2+\frac{a}{\gamma-1}\varrho^\gamma+\eta\ln\eta+\eta\Phi\right)\\ &+\partial_x\left(\frac{1}{2}\varrho u^3+\frac{a\gamma}{\gamma-1}\varrho^\gamma u+[(\ln\eta+1+\Phi)(\eta u-\eta\Phi_x-\partial_x\eta)]\right)\\ &\qquad-\varepsilon\partial_x\left(u\partial_x u\right)+\varepsilon|\partial_x u|^2+\left(\frac{\partial_x\eta}{\sqrt{\eta}}+\sqrt{\eta}\Phi_x\right)^2=-\beta\varrho u\Phi_x\end{aligned}$$

is obtained. Integrating over $\mathbb{R}$ and using the boundary conditions (1.3),

$$\begin{aligned}(3.11)\quad &\frac{d}{dt}\int_{\mathbb{R}}\frac{1}{2}\varrho|u|^2+\frac{a}{\gamma-1}\varrho^\gamma+\eta\ln\eta+\beta\varrho\Phi+\eta\Phi\ dx\\ &\qquad+\varepsilon\int_{\mathbb{R}}|\partial_x u|^2\ dx+\int_{\mathbb{R}}\left(\frac{\partial_x\eta}{\sqrt{\eta}}+\sqrt{\eta}\Phi_x\right)^2\ dx=-\int_{\mathbb{R}}\beta\varrho u\Phi_x\ dx.\end{aligned}$$

To bound the term on the right side of (3.11), it is noted by Young's inequality that

$$(3.12)\quad \left|\int_{\mathbb{R}}\beta\varrho u\Phi_x\ dx\right|\le\frac{1}{2}\int\varrho|u|^2\ dx+\frac{1}{2}\beta\int_{\mathbb{R}}|\Phi_x|\varrho\ dx.$$

Using that fact that Φ_x is compactly supported, that $\varrho\le C(1+e^*(\varrho,\overline{\varrho}))$ (see [12, 13, 17, 18], among others), and the control of the negative part of $\eta\ln\eta$ from the confinement hypotheses,

$$(3.13)\quad \frac{d}{dt}E(\varrho,u,\eta)+\varepsilon\int_{\mathbb{R}}|\partial_x u|^2\ dx+\int_{\mathbb{R}}\left(\frac{\partial_x\eta}{\sqrt{\eta}}+\sqrt{\eta}\Phi_x\right)^2\ dx\le C+CE.$$

Gronwall's inequality completes the proof. □

Lemma 3.1 in conjunction with Proposition 3.1 leads immediately to the following corollary, which allows for control of the $\eta\ln\eta$ term in $\mathcal{H}(\varrho,u|\overline{\varrho},0)$.

Corollary 3.1. *Let Φ satisfy the conditions in Definition 1.1. For any non-negative η in $L^1(\mathbb{R})$, if*

$$\int_{\mathbb{R}}\eta\ln\eta+\eta\Phi\ dx\le C$$

for some $C>0$, then $\eta\ln\eta\in L^1(\mathbb{R})$ and there exists some constant $D>0$ depending on C and Φ such that

$$\int_{\mathbb{R}}\eta\ln_+\eta\ dx\le D$$

and

$$\int_{\mathbb{R}}\eta\Phi\ dx\le D.$$

Proposition 3.1 also gives the following estimate on η, which follows from control of

$$\int_0^T\int_{\mathbb{R}}\left|\frac{\partial_x\eta}{\sqrt{\eta}}+\sqrt{\eta}\Phi_x\right|^2\ dx\ dt$$

and standard Sobolev inequalities.

Corollary 3.2. *Let ϱ, u, and η solve* (1.2)*. Then for each $T > 0$, it holds that $\eta \in L^2(0,T;C^{0,l}(\mathbb{R}))$ for some $l \in \left(0, \frac{1}{2}\right)$. Moreover, this norm is bounded independently of ε.*

Next, in the spirit of [12], integrability of $\partial_x \varrho$ is considered.

Lemma 3.2. *Let ϱ, u, η be smooth solutions to* (1.2), (1.3)*. Assume also that*

$$\left\| \sqrt{\varepsilon} \frac{\eta}{\varrho} \right\|_{L^2(0,T;L^2(\mathbb{R}))} \leq C \tag{3.14}$$

where C is independent of ε. Then if

$$\varepsilon^2 \int_{\mathbb{R}} \frac{|\partial_x \varrho_0|^2}{\varrho_0^3} \, dx < C_1 \text{ and } \varepsilon \int_{\mathbb{R}} \frac{\eta_0}{\varrho_0} \, dx \leq C_2$$

for some $C_1, C_2 > 0$, there is some finite constant $C > 0$ independent of ε such that

$$\begin{aligned} \varepsilon^2 \int_{\mathbb{R}} \frac{|\partial_x \varrho(x,T)|^2}{(\varrho(x,T))^3} \, dx + 2\varepsilon \int_{\mathbb{R}} \frac{\eta(x,T)}{\varrho(x,T)} \, dx& \\ + 2a\gamma\varepsilon \int_0^T \int_{\mathbb{R}} \varrho^{\gamma-3} |\partial_x \varrho|^2 \, dx \, dt &\leq C. \end{aligned} \tag{3.15}$$

Proof. This proposition is similar to [12, Lemma 3.2] and the proof here is follows the same spirit as the proof there. Letting $v := \frac{1}{\varrho}$, (1.2a) can be written as

$$v_t + uv_x = vu_x \tag{3.16}$$

which becomes after differentiating in x,

$$v_{xt} + (uv_x)_x = (vu_x)_x. \tag{3.17}$$

Multiplying by $2\varrho v_x$ and performing some straight-forward calculations yields

$$\left(\varrho |v_x|^2\right)_t + \left(\varrho u |v_x|^2\right)_x = 2v_x u_{xx}. \tag{3.18}$$

Solving for u_{xx} in (1.2b) and substituting, the right side of (3.18) becomes

$$\begin{aligned} 2v_x u_{xx} &= \frac{2}{\varepsilon} v_x [(a\varrho^\gamma)_x + (\varrho u)_t + (\varrho u^2)_x + \eta_x + (\beta\varrho + \eta)\Phi_x] \\ &= \frac{2}{\varepsilon} v_x (a\varrho^\gamma)_x + \frac{2}{\varepsilon}(\varrho u v_x)_t + \frac{2}{\varepsilon}[\varrho u (v_x u)_x - \varrho u (v u_x)_x + v_x (\varrho |u|^2)_x] \\ &\qquad + \frac{2}{\varepsilon}[v_x(\eta_x + (\beta\varrho + \eta)\Phi_x)] \end{aligned} \tag{3.19}$$

A simple calculation shows

$$v_x (a\varrho^\gamma)_x = -a\gamma \varrho^{\gamma-3} |\varrho_x|^2. \tag{3.20}$$

Integrating $\varrho u (v_x u)_x - \varrho u (v u_x)_x + v_x (\varrho |u|^2)_x$ over $\mathbb{R}$ yields, using integration by parts

$$\int_{\mathbb{R}} \varrho u (v_x u)_x - \varrho u (v u_x)_x + v_x (\varrho |u|^2)_x \, dx = \int_{\mathbb{R}} |\partial_x u|^2 \, dx. \tag{3.21}$$

Integrating (3.18) over $\mathbb{R}\times[0,T]$, using (3.19)-(3.21), and multiplying by ε^2 yield

$$\begin{aligned}
\varepsilon^2\int_{\mathbb{R}}\frac{|\partial_x\varrho(x,T)|^2}{(\varrho(x,T))^3}\,dx+2a\gamma\varepsilon\int_0^T\int_{\mathbb{R}}\varrho^{\gamma-3}|\partial_x\varrho|^2\,dx\,dt\\
=-2\varepsilon\int_{\mathbb{R}}\frac{\partial_x\varrho(x,T)u(x,T)}{\varrho(x,T)}\,dx+2\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x u|^2\,dx\,dt+2\varepsilon\int_{\mathbb{R}}\frac{(\partial_x\varrho_0)u_0}{\varrho_0}\,dx\\
+\varepsilon^2\int_{\mathbb{R}}\frac{|\partial_x\varrho_0|^2}{\varrho_0^3}\,dx+2\varepsilon\int_0^T\int_{\mathbb{R}}\partial_x\left(\frac{1}{\varrho}\right)[\partial_x\eta+(\beta\varrho+\eta)\Phi_x]\,dx\,dt.
\end{aligned}\tag{3.22}$$

It is clear using Young's inequality and (3.2) that

$$\begin{aligned}
&\left|2\varepsilon\int_{\mathbb{R}}\frac{\partial_x\varrho(x,T)u(x,T)}{\varrho(x,T)}\,dx\right|\\
&\leq\frac{\varepsilon^2}{4}\int_{\mathbb{R}}\frac{|\partial_x\varrho(x,T)|^2}{(\varrho(x,T))^3}\,dx+4\int_{\mathbb{R}}\varrho(x,T)|u(x,T)|^2\,dx\\
&\leq\frac{\varepsilon^2}{4}\int_{\mathbb{R}}\frac{|\partial_x\varrho(x,T)|^2}{(\varrho(x,T))^3}\,dx+C,
\end{aligned}\tag{3.23}$$

and similarly,

$$\begin{aligned}
\left|2\varepsilon\int_{\mathbb{R}}\frac{\partial_x\varrho_0u_0}{\varrho_0}\,dx\right|&\leq\frac{\varepsilon^2}{4}\int_{\mathbb{R}}\frac{|\partial_x\varrho_0|^2}{\varrho_0^3}\,dx+4\int_{\mathbb{R}}\varrho_0|u_0|^2\,dx\\
&\leq\frac{\varepsilon^2}{4}\int_{\mathbb{R}}\frac{|\partial_x\varrho_0|^2}{\varrho_0^3}\,dx+C.
\end{aligned}\tag{3.24}$$

Using Young's inequality, the bound $\varrho\leq C(1+e^*(\varrho,\overline{\varrho}))$, the compact support of Φ_x, and (3.2) it is clear that

$$\left|\int_0^T\int_{\mathbb{R}}\varepsilon\partial_x\left(\frac{1}{\varrho}\right)\beta\varrho\phi_x\,dx\,dt\right|\leq C+C\varepsilon^2\int_0^T\int_{\mathbb{R}}\frac{|\partial_x\varrho|^2}{\varrho^3}\,dx\,dt.\tag{3.25}$$

Next, control of

$$-2\varepsilon\int_0^T\int_{\mathbb{R}}\partial_x\left(\frac{1}{\varrho}\right)(\partial_x\eta+\eta\Phi_x)\,dx\,dt$$

is investigated. By integration by parts, this becomes

$$\begin{aligned}
2\varepsilon\int_0^T\int_{\mathbb{R}}\frac{1}{\varrho}\partial_x(\partial_x\eta+\eta\Phi_x)\,dx&=2\varepsilon\int_0^T\int_{\mathbb{R}}\frac{1}{\varrho}(\partial_t\eta+\partial_x(\eta u))\,dx\\
&=2\varepsilon\int_{\mathbb{R}}\frac{\eta(x,T)}{\varrho(x,T)}\,dx-2\varepsilon\int_{\mathbb{R}}\frac{\eta_0}{\varrho_0}\,dx-2\varepsilon\int_0^T\int_{\mathbb{R}}\frac{\eta}{\varrho}\partial_x u\,dx\,dt
\end{aligned}\tag{3.26}$$

where (3.16) has been employed to obtain the last equality. It is left to control the last integral above. Using Young's inequality and (3.14),

$$\begin{aligned}
\left|2\varepsilon\int_0^T\int_{\mathbb{R}}\frac{\eta}{\varrho}\partial_x u\,dx\,dt\right|&\leq\varepsilon\int_0^T\int_{\mathbb{R}}\frac{\eta^2}{\varrho^2}\,dx\,dt+\varepsilon\int_0^T\int_{\mathbb{R}}|\partial_x u|^2\,dx\,dt\\
&\leq C\varepsilon\int_0^T\int_{\mathbb{R}}\frac{\eta^2}{\varrho^2}\,dx\,dt+C\leq C
\end{aligned}\tag{3.27}$$

with the penultimate inequality following from (3.2).

□

Next, higher integrability of the fluid pressure is investigated

Lemma 3.3. *For smooth solutions to* (1.2) *with* $E_0 < \infty$ *independent of* ε, *for any compact subset* K *of* $\mathbb{R}$ *and for each* $T > 0$, *there is some constant* $C = C(K, E_0, T)$ *independent of* ε *such that*

$$\int_0^T \int_K \varrho(x,t)^{\gamma+1}\ dx\ dt \leq C. \tag{3.28}$$

Proof. This proof follows the spirit of the proof in [12, Lemma 3.3]. Let $\omega \in C_c^\infty(\mathbb{R})$ such that $\omega(x) \in [0,1]$ for any $x \in \mathbb{R}$. Multiplying (1.2b) by ω and integrating over $(-\infty, x)$ yields

$$\begin{aligned}\varrho u^2\omega + p(\varrho)\omega = \varepsilon\partial_x u\omega - \partial_t \int_{-\infty}^x \varrho u\omega\ dy + \int_{-\infty}^x (\varrho u^2 + p(\varrho) - \varepsilon\partial_x u)\omega_x\ dy \\ - \eta\omega + \int_{-\infty}^x \eta\omega_x - (\beta\varrho + \eta)\Phi_x\omega\ dy.\end{aligned} \tag{3.29}$$

Multiplying this by $\varrho\omega$ and using (1.2a) gives

$$\begin{aligned}p(\varrho)\varrho\omega^2 &= -\varrho^2u^2\omega^2 + \varepsilon\varrho\partial_x u\omega^2 - \partial_t\left(\varrho\omega\int_{-\infty}^x \varrho u\omega\ dy\right) \\ &- \partial_x\left(\varrho u\omega\int_{-\infty}^x \varrho u\omega\ dy\right) + \varrho u\omega_x\int_{-\infty}^x \varrho u\omega\ dy \\ &+ \varrho\omega\int_{-\infty}^x (\varrho u^2 + p(\varrho) - \varepsilon\partial_x u)\omega_x\ dy - \eta\varrho\omega^2 + \varrho\omega\int_{-\infty}^x \eta\omega_x - (\beta\varrho + \eta)\Phi_x\omega\ dy.\end{aligned} \tag{3.30}$$

Integrating (3.30) over $[0,T] \times \mathbb{R}$ yields

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}} \varrho p(\varrho)\omega^2\ dx\ dt \\ &= \varepsilon\int_0^T \int_{\mathbb{R}} \varrho\partial_x u\omega^2\ dx\ dt \\ &- \int_{\mathbb{R}} \varrho(x,t)\omega(x)\left(\int_{-\infty}^x \varrho(y,t)u(y,t)\omega(y)\ dy\right)\ dx\ dt \\ &\qquad + \int_{\mathbb{R}} \varrho_0\omega\left(\int_{-\infty}^x \varrho_0 u_0\omega\ dy\right)\ dx\ dt + r_1(T) + r_2(T)\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}r_1(T) = \int_0^T \int_{\mathbb{R}} \varrho\omega\left(\int_{-\infty}^x (\varrho u^2 + p(\varrho) - \varepsilon\partial_x u)\omega_x\ dy\right)\ dx\ dt \\ + \int_0^T \int_{\mathbb{R}} \varrho u\omega_x\left(\int_{-\infty}^x \varrho u\omega\ dy\right)\ dx\ dt\end{aligned}$$

and

$$r_2(T) = \int_0^T \int_{\mathbb{R}} \varrho\omega \left(\int_{-\infty}^x \eta\omega_x - (\beta\varrho + \eta)\Phi_x\omega \ dy \right) \ dx \ dt$$
$$- \int_0^T \int_{\mathbb{R}} \eta\varrho\omega^2 \ dx \ dt.$$

Noting that $\varrho^2\omega^2 \le \varrho^2 \le C(1+\varrho^{\gamma+1})$ since $\gamma > 1$, Young's inequality yields

$$\begin{aligned}(3.32)\quad \varepsilon \int_0^T \int_{\mathbb{R}} \varrho\partial_x u\omega^2 \ dx \ dt &\le \frac{\varepsilon^2}{\delta} \int_0^T \int_{\mathbb{R}} |\partial_x u|^2 \ dx \ dt + \delta \int_0^T \int_{\mathbb{R}} \varrho^2\omega^4 \ dx \ dt \\ &\le \frac{\varepsilon_0}{\delta}\varepsilon \int_0^T \int_{\mathbb{R}} |\partial_x u|^2 \ dx \ dt + C\delta \int_0^T \int_{\mathbb{R}} (1+\varrho^{\gamma+1})\omega^2 \ dx \ dt \\ &\le \frac{C}{\delta} + C_\omega\delta + C\delta \int^T \int_{\mathbb{R}} \varrho^{\gamma+1}\omega^2 \ dx \ dt.\end{aligned}$$

Noting that by Hölder's inequality and the the bound $\varrho \le C(1+e^*(\varrho,\overline{\varrho}))$

$$\left| \int_{-\infty}^x \varrho u\omega \ dy \right| \le \int_{K'} |\varrho u| \ dy$$
$$\le \left(\int_{K'} \varrho \ dy \right)^{\frac{1}{2}} \left(\int_{K'} \varrho u^2 \ dy \right)^{\frac{1}{2}} \le C$$

where $K' \stackrel{\text{def}}{=} \operatorname{supp}(\omega) \cap (-\infty, x)$, the estimate

$$(3.33)\quad \int_{\mathbb{R}} \varrho(x,t)\omega(x) \left(\int_{-\infty}^x \varrho u\omega \ dy \right) \ dx \le C$$

holds for any $t \in [0,T]$, using the compact support of ω. By similar arguments, $r_1(T)$ and the first integral of $r_2(T)$ are bounded by some C, and the second integral in $r_2(T)$ is non-negative, so combining (3.31)-(3.33) and taking δ small enough complete the proof. □

Next, (2.19) is used to help prove the following estimate.

Lemma 3.4. *Let ϱ_0 and u_0 be such that, in addition to the conditions in Proposition 3.1 and Lemma 3.2,*

$$(3.34)\quad \int_{\mathbb{R}} \varrho_0 u_0 \ dx \le M_0 < \infty$$

where M_0 is some constant independent of ε. Also assume that

$$\frac{\eta}{\varrho} \le C$$

for some constant C independent of ε for any $(x,t) \in \mathbb{R} \times [0,T]$. Then for any compact subset K of $\mathbb{R}$ and $T > 0$, there is a constant independent of ε such that

$$(3.35)\quad \int_0^T \int_K \varrho|u|^3 + \varrho^{\gamma+\theta} \ dx \ dt \le C.$$

Proof. Multiplying (1.2a) by $\partial_\varrho H^\#$, (1.2b) by $\partial_m H^\#$, and adding these equations together yields

$$\begin{aligned}\partial_t H^\#(\varrho,m)+\partial_x Q^\#(\varrho,m) &= -\partial_m H^\#(\varrho,m)[\partial_x\eta-(\beta\varrho+\eta)\Phi_x] \\ &\quad+\varepsilon\partial_x(\partial_m H^\#(\varrho,m)\partial_x u)-\varepsilon\partial_{mu}H^\#(\varrho,m)(\partial_x u)^2-\varepsilon\partial_{m\varrho}H^\#(\varrho,m).\end{aligned}\tag{3.36}$$

Integrating over $(0,T)\times(-\infty,x)$ gives

$$\begin{aligned}&\int_{-\infty}^x H^\#(\varrho(y,T),m(y,T))-H^\#(\varrho_0(y),m_0(y))\;dy \\ &+\int_0^T Q^\#(\varrho(x,t),m(x,t))\;dt \\ &=-\int_0^T\int_{-\infty}^x \partial_m H^\#(\varrho,m)\partial_x\eta\;dy\;dt \\ &-\int_0^T\int_{-\infty}^x \partial_m H^\#(\varrho,m)(\beta\varrho+\eta)\Phi_x\;dy\;dt \\ &+\varepsilon\int_0^T \partial_m H^\#(\varrho(x,t),m(x,t))\partial_x u(x,t)\;dt \\ &-\varepsilon\int_0^T\int_{-\infty}^x \partial_{m\varrho}H^\#(\varrho,m)\partial_x\varrho\cdot\partial_x u\;dy\;dt \\ &\qquad\qquad-\varepsilon\int_0^T\int_{-\infty}^x \partial_{mu}H^\#(\varrho,m)|\partial_x u|^2\;dy\;dt+\overline{Q},\end{aligned}\tag{3.37}$$

where $\overline{Q}\overset{\text{def}}{=}Q^\#(\overline{\varrho},0)T$. It follows from Proposition 3.1 and Lemma 2.2 that

$$\left|\varepsilon\int_0^T\int_{-\infty}^x \partial_{mu}H^\#(\varrho,m)|\partial_x u|^2\;dy\;dt\right|\le C.\tag{3.38}$$

Using Young's inequality and Lemma 2.2,

$$\begin{aligned}&\left|\varepsilon\int_0^T\int_{-\infty}^x \partial_{m\varrho}H^\#(\varrho,m)\partial_x\varrho\partial_x u\;dy\;dt\right| \\ &\qquad\qquad\le C\varepsilon\int_0^T\int_{\mathbb{R}}\varrho^{\gamma-3}|\partial_x\varrho|^2+|\partial_x u|^2\;dx\;dt,\end{aligned}\tag{3.39}$$

which is bounded by a constant using Proposition 3.1 and Lemma 3.2. Using Lemma 2.2 and Young's inequality,

$$\begin{aligned}\left|\varepsilon\int_0^T \partial_m H^\#\partial_x u\;dt\right| &\le C\varepsilon\int_0^T |u||\partial_x u|+\varrho^\theta|\partial_x u|\;dt \\ &\le C\varepsilon\int_0^T |u|^2+\varrho^{\gamma-1}+|\partial_x u|^2\;dt.\end{aligned}\tag{3.40}$$

Integrating (3.37) over the compact set $K \subset \mathbb{R}$ and using the bounds (3.38)-(3.40) and using the lower bound for $Q^{\#}$ from Lemma 2.2 yields

$$
\begin{aligned}
(3.41)\quad \int_0^T \int_K \varrho|u|^3 + \varrho^{\gamma+\theta}\, dx\, dt &\le \int_0^T \int_K Q^{\#}\, dx\, dt \\
&\le -\int_K \int_{-\infty}^x H^{\#}(\varrho(y,T), m(y,T)) - H^{\#}(\varrho_0(y), m_0(y))\, dy\, dx \\
&\quad - \int_0^T \int_K \int_{-\infty}^x \partial_m H^{\#}(\varrho, m)[\beta\varrho\Phi_x + (\partial_x\eta + \eta\Phi_x)]\, dy\, dx\, dt \\
&\qquad + C\varepsilon \int_0^T \int_K |u|^2 + \varrho^{\gamma-1} + |\partial_x u|^2\, dx\, dt + C_K
\end{aligned}
$$

where C_K is a constant depending on the compact set K.

Invoking Lemma 2.2,

$$
(3.42)\quad \left| \int_K \int_{-\infty}^x H^{\#}(\varrho(y,T), m(y,T)) - H^{\#}(\varrho_0(y), m_0(y))\, dy\, dx \right| \le C_K.
$$

Using Proposition 3.1 and Lemma 3.2,

$$
(3.43)\quad C\varepsilon \int_0^T \int_K \varrho^{\gamma-1} + |\partial_x u|^2\, dx\, dt \le C_K.
$$

To handle the $|u|^2$ term in the last integral in (3.41), it is noted that the set

$$
B(t) \stackrel{\text{def}}{=} \left\{ x \in \mathbb{R} : \varrho(x,t) \le \frac{\overline{\varrho}}{2} \right\}
$$

has measure

$$
(3.44)\quad |B(t)| \le \frac{C(t)}{e^*\left(\frac{\overline{\varrho}}{2}, \overline{\varrho}\right)}
$$

by using Proposition 3.1 to obtain

$$
(3.45)\quad \int_{\varrho(x,t)\le\overline{\varrho}/2} e^*(\varrho, \overline{\varrho})\, dx \le C(t)
$$

for some non-decreasing function $C(t) > 0$. For any interval $[a,b] \subset K$ such that $b - a = \frac{2C(t)}{e^*(\overline{\varrho}/2, \overline{\varrho})}$, there is some measurable $A(t) \subset (a,b)$ such that

$$
|A(t)| \le \frac{C(t)}{e^*\left(\frac{\overline{\varrho}}{2}, \overline{\varrho}\right)}.
$$

Defining $u_A(t)$ to be the average value of $u(x,t)$ on the set A, it is clear that for any $x \in [a,b]$ that

$$
(3.46)\quad |u(x,t)| \le |u_A(t)| + \int_K |\partial_x u|\, dx.
$$

Using a simple Hölder's inequality argument and Proposition 3.1 yields that

$$
|u_A(t)| \le C_K,
$$

so

$$
(3.47)\quad \varepsilon \int_0^T \int_K |u|^2\, dx\, dt \le C_K.
$$

To control the $\partial_m H^{\#}(\varrho, m)\beta\varrho\Phi_x$ term of the second integral on the right side of (3.41), let $K' = (-\infty, x) \cap \operatorname{supp} \Phi_x$. Then using Lemma 2.2,

$$\left|\int_0^T \int_K \int_{-\infty}^x \partial_m H^{\#}(\varrho, m)\beta\varrho\Phi_x \, dy \, dx \, dt\right| \leq C \int_0^T \int_K \int_{K'} \varrho|u| + \varrho^{(\gamma+1)/2} \, dy \, dx \, dt \leq C \int_0^T \int_K \int_{K'} \varrho + \varrho|u|^2 + 1 + \varrho^{\gamma+1} \, dy \, dx \, dt \leq C_K \tag{3.48}$$

using Lemma 3.3 and the techniques therein, and by Proposition 3.1.

Now, it is left to bound the term

$$-\int_0^T \int_K \int_{-\infty}^x \partial_m H^{\#}(\varrho, m)[\partial_x \eta + \eta\Phi_x] \, dy \, dx \, dt.$$

This value is bounded by

$$C \int_0^T \int_{\mathbb{R}} (|u| + \varrho^\theta)\sqrt{\varrho}\sqrt{\frac{\eta}{\varrho}} \left|\frac{\partial_x \eta}{\sqrt{\eta}} + \sqrt{\eta}\Phi_x\right| \, dx \, dt \leq C \int_0^T \int_{\mathbb{R}} \varrho|u|^2 + e^*(\varrho, \overline{\varrho}) + \frac{\eta}{\varrho}\left|\frac{\partial_x \eta}{\sqrt{\eta}} + \sqrt{\eta}\Phi_x\right|^2 \, dx \, dt \leq C. \tag{3.49}$$

the last inequality following immediately from Proposition 3.1 and that

$$\frac{\eta}{\varrho} \leq C$$

for some positive constant C independent of ε. □

4. Compactness of Weak Entopy Measures

Next, the H^{-1} compactness of weak entropy dissipation measures for (1.2) is considered using the estimates from the previous section. The entropies considered are those induced by functions $\psi \in C_c^2(\mathbb{R})$ in the sense of (2.10) and (2.11).

Proposition 4.1. *Fix $\psi \in C_c^2(\mathbb{R})$ and let H^ψ and Q^ψ be as defined in* (2.10) *and* (2.11)*, respectively. For each fixed $\varepsilon \in (0, \varepsilon_0]$, let $(\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon)$ solve* (1.2)*. Then the sequence of entropy dissipation measures*

$$\left\{\partial_t H^\psi(\varrho_\varepsilon, u_\varepsilon) + \partial_x Q^\psi(\varrho_\varepsilon, u_\varepsilon)\right\}_{\varepsilon \in (0, \varepsilon_0]} \tag{4.1}$$

is contained in a compact subset of $H^{-1}_{\mathrm{loc}}([0, \infty) \times \mathbb{R})$.

Proof. For clarity of notation, the subscripts of ϱ_ε, u_ε, and η_ε are omitted. Multiplying (1.2a) by $\partial_\varrho H^\psi(\varrho, m)$ and (1.2b) by $\partial_m H^\psi(\varrho, m)$ and adding the results together gives

$$\begin{aligned} \partial_t H^\psi + \partial_x Q^\psi &= -\partial_m H^\psi \partial_x \eta - \partial_m H^\psi(\beta\varrho + \eta)\Phi_x + \varepsilon\partial_m H^\psi \partial_{xx} u \\ &= -\partial_m H^\psi(\partial_x \eta + \eta\Phi_x) - \beta\partial_m H^\psi \varrho\Phi_x + \varepsilon\partial_x\left(\partial_m H^\psi \partial_x u\right) \\ &\quad - \varepsilon\partial_{mu} H^\psi |\partial_x u|^2 - \varepsilon\partial_{m\varrho} H^\psi \partial_x \varrho \partial_x u. \end{aligned} \tag{4.2}$$

Using Young's inequality, Corollary 3.2, Proposition 3.1 and Lemma 2.1,

$$\left|\int_0^T \int_{\mathbb{R}} \partial_m H^\psi(\partial_x + \eta\Phi_x)\ dx\ dt\right| \leq C_\psi \int_0^T \int_{\mathbb{R}} \eta + \left|\frac{\partial_x}{\sqrt{\eta}} + \sqrt{\eta}\Phi_x\right|\ dx\ dt \leq C_\psi. \tag{4.3}$$

Therefore, $\{\partial_m H^\psi(\partial_x\eta + \eta\Phi_x)\}_{\varepsilon\in(0,\varepsilon_0]}$ is uniformly bounded in ε in $L^1([0,T]\times\mathbb{R})$, so it is compact in $W^{-1,q_1}_{\text{loc}}([0,\infty)\times\mathbb{R})$ for $q_1 \in (1,2)$.

Using the bound $\varrho \leq C(1+e^*(\varrho,\overline{\varrho}))$ and the bound on $\partial_m H^\psi \leq C_\psi$, it is clear using the compact support and bounds of Φ_x and using Proposition 3.1 that

$$\left\|\beta\partial_m H^\psi \varrho\Phi_x\right\|_{L^1([0,T]\times\mathbb{R})} \leq C_\psi, \tag{4.4}$$

so $\left\{\beta\partial_m H^\psi\varrho\Phi_x\right\}_{\varepsilon\in(0,\varepsilon_0]}$ is compact in $W^{-1,q_1}_{\text{loc}}([0,\infty)\times\mathbb{R})$ for any $q_1 \in (1,2)$. By Lemma 2.1, Proposition 3.1, Young's Inequality, and Lemma 3.2,

$$\begin{aligned}\varepsilon\int_0^T \int_{\mathbb{R}} &\partial_{mu}H^\psi|\partial_x u|^2 + \partial_{m\varrho}\partial_x\varrho\partial_x u\ dx\ dt \\ &\leq \varepsilon C_\psi \int_0^T \int_{\mathbb{R}} |\partial_x u|^2 + \varrho^{\theta-1}\partial_x\varrho\partial_x u\ dx\ dt \\ &\leq C_\psi + \varepsilon C_\psi \int_0^T \int_{\mathbb{R}} \varrho^{\gamma-3}|\partial_x\varrho|^2 + |\partial_x u|^2\ dx\ dt \leq C_\psi.\end{aligned} \tag{4.5}$$

Thus,

$$\left\{-\varepsilon\partial_{mu}H^\psi|\partial_x u|^2 - \varepsilon\partial_{m\varrho}H^\psi\partial_x\varrho\partial_x u\right\}_{\varepsilon\in(0,\varepsilon_0]}$$

is uniformly bounded in ε in $L^1([0,T]\times\mathbb{R})$, which means it is contained in a compact subset of $W^{-1,q_1}_{\text{loc}}([0,\infty)\times\mathbb{R})$ for any $q_1 \in (1,2)$.

By Lemma 2.1, $\left|\partial_m H^\psi(\varrho,m)\right| \leq C_\psi$, so using Proposition 3.1,

$$\varepsilon^2\int_0^T \int_{\mathbb{R}} \left|\partial_m H^\psi\right|^2 |\partial_x u|^2\ dx\ dt \leq C_\psi\varepsilon \to 0, \tag{4.6}$$

which means $\varepsilon\partial_m H^\psi\partial_x u \to 0$ in $L^2([0,T]\times\mathbb{R})$. Therefore, $\partial_x\left(\varepsilon\partial_m H^\psi\partial_x u\right) \to 0$ in $W^{-1,2}([0,T]\times\mathbb{R})$, and in particular, in $W^{-1,q_1}_{\text{loc}}([0,\infty)\times\mathbb{R})$ for any $q_1 \in (1,2)$. From (4.2) and combining the results from (4.3)-(4.6), it is clear that

$$\left\{\partial_t H^\psi + \partial_x Q^\psi\right\}_{\varepsilon\in(0,\varepsilon_0]} \tag{4.7}$$

is contained in a compact set in $W^{-1,q_1}_{\text{loc}}([0,\infty)\times\mathbb{R})$ for any $q_1 \in (1,2)$.

For $\gamma \in (1,3]$, Lemma 2.1 gives that H^ψ and Q^ψ are bounded by $C_\psi\varrho$. Fixing a compact set $K \subset \mathbb{R}$, Lemma 3.3 makes it clear that

$$\left\{\left|H^\psi\right| + \left|Q^\psi\right|\right\}_{\varepsilon\in(0,\varepsilon_0]} \tag{4.8}$$

is bounded in ε in $L^{\gamma+1}([0,T]\times K)$. Thus

$$\partial_t H^\psi + \partial_x Q^\psi$$

is uniformly bounded in $W^{-1,\gamma+1}([0,\infty)\times K)$ in ε. □

5. Young Measure

To start this section, a somewhat classical definition of Young measure is given (see [16]).

Definition 5.1. Let $\{f_\varepsilon\}$ be a bounded sequence in $L^\infty(\Omega;\mathbb{R}^n)$ where Ω is a domain in $\mathbb{R}^m$. Then there exists a subsequence (not relabeled) $\{f_\varepsilon\}$ and a probability measure ν_y for almost every $y \in \Omega$ on $\mathbb{R}^n$ such that for each $\phi \in C(\mathbb{R}^n)$, $\phi(f_\varepsilon)$ converges to $\int_\Omega \phi(y)\, d\nu_y$ weakly-$*$ in $L^\infty(\Omega)$. The measures ν_y are called the Young measures generated by $\{f_\varepsilon\}$.

The existence of these Young measures is a fairly classical result at this point. For a proof, see [16, Theorem 11].

Remark 5.1. At this point onward to avoid confusion, $(\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon)$ represent solutions to (1.2) and (ϱ, u, η) are the limits as $\varepsilon \to 0$.

In this section, the Young measures associated with the sequence of solutions $(\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon)$ of (1.2) are considered. Specifically, $\nu_{x,t}$ is a Young measure corresponding to the sequence $(\varrho_\varepsilon, u_\varepsilon)$ from the sequence of solutions to (1.2). The Young measures are derived from the fluid density and fluid velocity and not with respect to the particle density due to the well-known difficulties in developing entropies for systems of more than two unknowns. Following the techniques in [19] which are used in [12, Section 5], define the space

$$\mathbb{H} \stackrel{\text{def}}{=} \{(\varrho, u) : \varrho > 0\}$$

and let $\overline{\mathcal{H}}$ be the compactification of $\mathbb{H}$ with $C(\overline{\mathcal{H}})$ isometrically isomorphic to $\overline{C}(\mathbb{H})$, the set of all $\phi \in C(\mathbb{H})$ such that $\phi(0,u)$ is constant and the function $(\varrho, u) \mapsto \lim_{s\to\infty} \phi(s\varrho, su)$ is a continuous function on the intersection of $\overline{\mathbb{H}}$ and $\mathbb{S}^1$.

In light of the above spaces, the Young measures considered in the current work will obey the following definition from [1], which uses the work in [4, 22].

Definition 5.2 (Young Measure)**.** Let $(\varrho_\varepsilon, u_\varepsilon)$ be a sequence of functions from $\mathbb{R}\times(0,\infty)$ to $\mathbb{H}$. The Young measure is the measure $\nu_{x,t} \in L^\infty_{\mathrm{w}}(\mathbb{R}\times(0,\infty); \mathrm{Prob}(\overline{\mathcal{H}}))$ such there is a subsequence (not relabeled) $(\varrho_\varepsilon, u_\varepsilon)$ such that for all $\phi \in C(\overline{\mathcal{H}})$,

$$\phi(\varrho_\varepsilon(x,t), u_\varepsilon(x,t)) \to \int_{\overline{\mathcal{H}}} \phi(\varrho, u)\ d\nu_{x,t}(\varrho, u) \tag{5.1}$$

weakly-$*$ in $L^\infty(\mathbb{R}\times(0,\infty))$. The sequence $(\varrho_\varepsilon, u_\varepsilon)$ converges to $(\varrho, u) : \mathbb{R}\times(0,\infty) \to \overline{\mathcal{H}}$ if and only if $\nu_{x,t} = \delta_{(\varrho(x,t), m(x,t))}$ for almost all (x,t).

The following propositions extend the class of test functions for the Young measure $\nu_{x,t}$ to a set larger than $\overline{C}(\mathbb{H})$. The first is proven using the bounds from Lemma 3.4 and the Lebesgue dominated convergence theorem and follows from the proof from [12, Proposition 5.1(i)].

Proposition 5.1. *For the Young measure $\nu_{x,t}$,*

$$\int_{\mathbb{H}} \varrho^{\gamma+1} + \varrho|u|^3\ d\nu_{x,t} \in L^1_{\mathrm{loc}}(\mathbb{R}\times(0,\infty)). \tag{5.2}$$

Next, the class of test functions is expanded as follows.

Proposition 5.2. *Let $\phi : (0,\infty)\times\mathbb{R} \mapsto \mathbb{R}$ such that*

(1) *$\phi \in C(\overline{\mathbb{H}})$ and $\phi \equiv 0$ on $\partial\mathbb{H}$,*

(2) *there is some* $a > 0$ *such that*

$$\operatorname{supp}\phi \subset \left\{(\varrho, u) : \varrho^\theta + u \geq -a \text{ and } u - \varrho^\theta \leq a\right\},$$

and

(3) $|\phi(\varrho, u)| \leq \varrho^{\kappa(\gamma+1)}$ *for some* $\kappa \in \left(0, \frac{1}{2}\right)$.

Then ϕ *is integrable over* $\mathbb{H}$ *with respect to the measure* $\nu_{x,t}$ *and*

$$\phi(\varrho_\varepsilon, u_\varepsilon) \to \int_{\mathbb{H}} \phi(\varrho, u)\, d\nu_{x,t} \tag{5.3}$$

weakly in $L^1_{\mathrm{loc}}((0,\infty) \times \mathbb{R})$.

Remark 5.2. The condition on κ above differs from that in [12, Proposition 5.1(ii)] in order that Lemma 5.1 can be proven. The corresponding claim in the proof of [12, Proposition 5.1(ii)] does not go through for κ close to one.

The following lemma is used in the proof of Proposition 5.2.

Lemma 5.1. *Let* K *be a compact subset of* $\mathbb{R}$ *and let* $\omega_k(\varrho, u)$ *be a non-negative, smooth function such that* $\omega_k(\varrho, u) = 1$ *on the set*

$$\left\{(\varrho, u) : \varrho^\theta \in \left[\frac{1}{k}, k\right],\ |u| \leq k\right\}$$

and such that $\omega_k(\varrho, u) = 0$ *outside the set*

$$\left\{(\varrho, u) : \varrho^\theta \in \left[\frac{1}{2k}, 2k\right],\ |u| \leq 2k\right\}.$$

Then for ϕ *meeting the hypotheses of Proposition 5.2,*

$$\lim_{k\to\infty} \int_{K\times[0,T]} \int_{\mathbb{H}} (\phi\omega_k)(\varrho_\varepsilon, u_\varepsilon)\, d\nu_{x,t}\, dx\, dt = \int_{K\times[0,T]} \int_{\mathbb{H}} \phi(\varrho_\varepsilon, u_\varepsilon)\, d\nu_{x,t}\, dx\, dt. \tag{5.4}$$

Moreover, this convergence is uniform in ε.

Remark 5.3. This is the claim in the proof of Proposition 5.2 in [12]. A clearer presentation of its proof is presented here.

Proof. Let $k_1 < k_2$. It is clear that $\omega_{k_1} - \omega_{k_2} = 0$ for

$$(\varrho, u) \in \left(\left[\frac{1}{k_1}, k_1\right] \times [-k_1, k_1]\right) \cup \left(\left[\frac{1}{2k_2}, 2k_2\right] \times [-2k_2, 2k_2]\right)^c$$

and

$$\sup_{0 \leq \varrho^\theta \leq k_1^{-1}} |\phi(\varrho, u)| \leq k_1^{-\kappa(\gamma+1)/\theta} \stackrel{\text{def}}{=} c_{k_1},$$

which goes to zero as $k_1 \to \infty$.

For

$$(\varrho, u) \in \operatorname{supp}\phi \cap \left\{\frac{1}{k_1} \leq \varrho^\theta \leq k_1,\ |u| \leq k_1\right\}^c,$$

either $\varrho^\theta \leq k_1^{-1}$ or $\varrho^\theta \geq k_1 - a$ provided $k_1 - a \geq k_1^{-1}$.

Fixing $\alpha > 0$, Young's inequality shows that for ϱ such that $\varrho^\theta \geq k_1 - a$,

$$|\phi(\varrho, u)| \leq C(\kappa, \alpha) + \alpha\varrho^{\gamma+1} \tag{5.5}$$

for any $\alpha > 0$. Then,

$$
\begin{aligned}
(5.6) \qquad & \left| \int_{[0,T]\times K} (\phi(\omega_{k_1} - \omega_{k_2})(\varrho_\varepsilon, u_\varepsilon)\ dx\ dt \right| \\
& \le T|K|c_{k_1} + C(\kappa,\alpha)|([0,T]\times K) \cap \{(x,t) : \varrho_\varepsilon^\theta > k_1 - a\})| \\
& \qquad\qquad + \alpha \int_{[0,T]\times K} \varrho_\varepsilon^{\gamma+1}\ dx\ dt.
\end{aligned}
$$

Using Chebyshev's inequality and Lemma 3.3,

$$
\begin{aligned}
(5.7) \qquad \left| \int_{[0,T]\times K} (\phi(\omega_{k_1} - \omega_{k_2})(\varrho_\varepsilon, u_\varepsilon)\ dx\ dt \right| & \le T|K|c_{k_1} + C(\kappa,\alpha) k_1^{-(\gamma+1)/\theta} + C\alpha \\
& \to C\alpha
\end{aligned}
$$

as $k_1 \to \infty$. Since $\alpha > 0$ is arbitrary, this proves the lemma. □

Proof of Proposition 5.2. Let K be a compact subset of $\mathbb{R}$. Then using Proposition 5.1 and the dominated convergence theorem, for almost all $(x,t) \in K \times [0,T]$,

$$
(5.8) \qquad \lim_{k\to\infty} \int_{\mathbb{H}} \phi\omega_k\ d\nu_{x,t} = \int_{\mathbb{H}} \phi\ d\nu_{x,t}
$$

and

$$
(5.9) \qquad \lim_{k\to\infty} \int_{K\times[0,T]} \int_{\mathbb{H}} \phi\omega_k\ d\nu_{x,t} = \int_{K\times[0,T]} \int_{\mathbb{H}} \phi\ d\nu_{x,t}.
$$

Using dominated convergence, the definition of Young measure, and (5.9),

$$
(5.10) \qquad \lim_{k\to\infty} \lim_{\varepsilon\to 0} \int_{K\times[0,T]} (\phi\omega_k)(\varrho_\varepsilon, u_\varepsilon)\ dx\ dt = \int_{K\times[0,T]} \int_{\mathbb{H}} \phi\ d\nu_{x,t}\ dx\ dt.
$$

Using Lemma 5.1 and dominated convergence arguments,

$$
\begin{aligned}
(5.11) \qquad & \lim_{\varepsilon\to 0} \int_{[0,T]\times K} \phi(\varrho_\varepsilon, u_\varepsilon)\ dx\ dt \\
& = \lim_{\varepsilon\to 0} \lim_{k\to\infty} \int_{[0,T]\times K} (\phi\omega_k)(\varrho_\varepsilon, u_\varepsilon)\ dx\ dt \\
& = \lim_{k\to\infty} \lim_{\varepsilon\to 0} \int_{[0,T]\times K} (\phi\omega_k)(\varrho_\varepsilon, u_\varepsilon)\ dx\ dt \\
& \qquad = \lim_{k\to\infty} \int_{[0,T]\times K} \int_{\mathbb{H}} \phi\omega_k\ d\nu_{x,t} = \int_{[0,T]\times K} \int_{\mathbb{H}} \phi\omega\ d\nu_{x,t},
\end{aligned}
$$

proving the proposition. □

Next, it is remarked that the Young measure $\nu_{x,t}$ is concentrated in $\mathbb{H}$ and the vacuum state $\varrho = 0$. This follows from the definition of the Young measure, the definition of $\mathcal{H}$, and the uniform bounds from Section 3.

Lemma 5.2. *Consider $\nu_{x,t}$ as an element of $(C(\overline{\mathcal{H}}))^*$. Then*

$$
(5.12) \qquad \nu_{x,t}[\overline{\mathcal{H}} \setminus (\mathbb{H} \cup \{\varrho = 0\})] = 0.
$$

6. Measure-Valued Solutions

In this section, the convergence of solutions $(\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon)$ of (1.2) to measure-valued solutions (ϱ, u, η) to (1.1) is investigated. This section begins with the definition of measure-valued solutions for (1.1).

Definition 6.1 (Measure-Valued Solutions)**.** Let $\{\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon\}_{\varepsilon\in(0,\varepsilon_0]}$ be a sequence of solutions to (1.2). Then $\nu_{x,t}$ is a measure-valued solution to (1.1) if and only if $\eta_\varepsilon \to \eta$ weakly and for any function of s $\psi \in \{\pm 1,\ \pm s,\ s^2\}$,

$$\partial_t \langle \nu_{x,t}, H^\psi\rangle + \partial_x \langle \nu_{x,t}, Q^\psi\rangle \le -\langle \nu_{x,t}, \partial_m H^\psi [\partial_x \eta + (\beta\varrho + \eta)\Phi_x]\rangle \tag{6.1}$$

and

$$\langle \nu_{x,t}, H^\psi\rangle(\cdot, 0) = H^\psi(\varrho_0, m_0) \tag{6.2}$$

in the sense of distributions.

This definition and the work above lead to the following proposition.

Proposition 6.1. *The Young measure $\nu_{x,t}$ derived from the solutions $\{\varrho_\varepsilon, u_\varepsilon, \eta_\varepsilon\}$ of* (1.2) *is a measure-valued solution to* (1.1)*.*

Proof. Using H^ψ and Q^ψ in (4.2), the equations (2.10) and (2.11) yield

$$\begin{aligned} &\partial_t H^\psi(\varrho_\varepsilon, m_\varepsilon) + \partial_x Q^\psi(\varrho_\varepsilon, m_\varepsilon) \\ &\quad= \varepsilon\partial_x \left[\partial_m H^\psi(\varrho_\varepsilon, u_\varepsilon)\partial_x u_\varepsilon\right] - \partial_m H^\psi(\partial_x\eta + \eta\Phi_x) - \beta\partial_m H^\psi \varrho\Phi_x \\ &\qquad - \int_{-1}^{1} \psi''\left(\frac{m_\varepsilon}{\varrho_\varepsilon} + s\varrho_\varepsilon^\theta\right)(1-s^2)^\lambda \left[\varepsilon|\partial_x u_\varepsilon|^2 + \varepsilon\theta\varrho_\varepsilon^\theta s\partial_x\varrho_\varepsilon\partial_x u_\varepsilon\right]\, ds. \end{aligned} \tag{6.3}$$

Since $\psi \in \{\pm 1,\ \pm s,\ s^2\}$, $\psi''(s) \ge 0$ for $s \in [-1, 1]$, which means (6.1) holds after taking $\varepsilon \to 0$ in (6.3). $\square$

The next step is to explore the commutator relation for H^ψ and Q^ψ. For ease of notation, define the entropy kernel as

$$\chi(\xi) \stackrel{\text{def}}{=} \left[\varrho^{2\theta} - (u-\xi)^2\right]_+^\lambda$$

and the define for any function $f(\varrho, u)$ such that

$$|f(\varrho, u)| \le \varrho|u|^3 + \varrho^{\gamma+\max\{1,\theta\}}$$

the weak limit of $f(\varrho_\varepsilon, u_\varepsilon)$ as

$$f(\varrho_\varepsilon, u_\varepsilon) \rightharpoonup \overline{f(\varrho, u)}(x,t) \stackrel{\text{def}}{=} \langle \nu_{x,t}, f(\varrho, u)\rangle.$$

Using the uniform bounds from Section 3 and the div-curl lemma, for any C_c^2 functions ϕ and ψ, $H^\psi Q^\phi - H^\phi Q^\psi$ is weakly continuous with respect to the weakly convergent sequence $(\varrho_\varepsilon, u_\varepsilon) \rightharpoonup (\varrho, u)$ and

$$\begin{aligned} &H^\psi(\varrho_\varepsilon, u_\varepsilon)Q^\phi(\varrho_\varepsilon, u_\varepsilon) - H^\phi(\varrho_\varepsilon, u_\varepsilon)Q^\psi(\varrho_\varepsilon, u_\varepsilon) \\ &\qquad\qquad \rightharpoonup \overline{H^\psi(\varrho, u)Q^\phi(\varrho, u)} - \overline{H^\phi(\varrho, u)Q^\psi(\varrho, u)} \end{aligned} \tag{6.4}$$

which yields the Tartar-Murat commutator relation

$$\begin{aligned} &\overline{H^\psi(\varrho, u)Q^\phi(\varrho, u) - H^\phi(\varrho, u)Q^\psi(\varrho, u)} \\ &\qquad\qquad = \overline{H^\psi(\varrho, u)Q^\phi(\varrho, u)} - \overline{H^\phi(\varrho, u)Q^\psi(\varrho, u)}. \end{aligned} \tag{6.5}$$

Using the definitions of the entropy/entropy-flux pairs based off the arbitrary continuous functions ψ and ϕ, the above becomes

$$
\begin{aligned}
(6.6)\qquad & \int_{\mathbb{R}^2} \psi(s_1)\phi(s_2)\overline{\chi(s_1)[\theta s_2 + (1-\theta)u]\chi(s_2)}\; ds_1\; ds_2 \\
&- \int_{\mathbb{R}^2} \psi(s_2)\phi(s_1)\overline{\chi(s_1)[\theta s_1 + (1-\theta)u]\chi(s_2)}\; ds_1\; ds_2 \\
&= \int_{\mathbb{R}} \psi(s_1)\overline{\chi(s_1)}\; ds_1 \int_{\mathbb{R}} \phi(s_2)\overline{[\theta s_2 + (1-\theta)u]\chi(s_2)}\; ds_2 \\
&- \int_{\mathbb{R}} \psi(s_2)\overline{\chi(s_2)}\; ds_2 \int_{\mathbb{R}} \phi(s_1)\overline{[\theta s_1 + (1-\theta)u]\chi(s_1)}\; ds_1 \\
&= \int_{\mathbb{R}^2} \psi(s_1)\phi(s_2)\overline{\chi(s_1)}\;\overline{[\theta s_2 + (1-\theta)u]\chi(s_2)}\; ds_1\; ds_2 \\
&\qquad\qquad - \int_{\mathbb{R}^2} \psi(s_2)\phi(s_1)\overline{\chi(s_2)}\;\overline{[\theta s_1 + (1-\theta)u]\chi(s_1)}\; ds_1\; ds_2.
\end{aligned}
$$

Letting $\phi \equiv \psi$ and noting that ψ is arbitrary, it is clear that

$$
\begin{aligned}
(6.7)\qquad & \overline{\chi(s_1)}\;\overline{[\theta s_2 + (1-\theta)u]\chi(s_2)} - \overline{\chi(s_2)}\;\overline{[\theta s_1 + (1-\theta)u]\chi(s_1)} \\
&\qquad\qquad = \theta(s_2 - s_1)\overline{\chi(s_1)\chi(s_2)}
\end{aligned}
$$

which yields the following result.

Lemma 6.1. *The measure-valued solution $\nu_{x,t}$ of (1.1) obeys the following commutator relation: for almost all s_1, $s_2 \in \mathbb{R}$,*

$$
\begin{aligned}
(6.8)\qquad & \theta(s_2 - s_1)[\overline{\chi(s_1)\chi(s_2)} - \overline{\chi(s_1)}\;\overline{\chi(s_2)}] \\
&\qquad\qquad = (1-\theta)[\overline{u\chi(s_2)}\;\overline{\chi(s_1)} - \overline{u\chi(s_1)}\;\overline{\chi(s_2)}].
\end{aligned}
$$

7. Reduction of the Measure-Valued Solutions

In this section, the measure-valued solution $\nu_{x,t}$ is a delta measure in the coordinates (ϱ, u) almost everywhere for (x, t). The argument is broken into two cases: $\gamma > 3$ and $\gamma \in (1, 3]$. It follows the compensated compactness framework from [12], also used in [13] and relies upon results on the commutator of the entropy kernel developed in [20].

7.1. **Reduction Large Adiabatic Constant.** First, the case where $\gamma > 3$ is considered. To begin with, the following result for the weak limit of the entropy kernel is presented which allows for the passage of the weak limit done in the proof of the measure reduction.

Lemma 7.1. *If $\gamma > 3$, then for $p \in \left[1, \frac{\gamma-1}{\gamma-3}\right)$,*

$$\overline{\chi(s)} \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}; L^p(\mathbb{R})).$$

This calculation is done in [12, Lemma 6.1] and is omitted here.

Next, let A be the open set that is the union of intervals of the form

$$(u - \varrho^\theta, u + \varrho^\theta)$$

for (ϱ, u) in the support of $\nu_{x,t}$, and let J be a connected component of A.

Proposition 7.1. *Any connected component J of A is bounded.*

Proof. Assume by contradiction, first, that J is not bounded below. Since J is unbounded below, fix $M_0 < 0$ such that $M_0 + 1 \in J$ and let $s_2 \in (M_0, M_0 + 1)$. Let $s_1 \leq -2|M_0|$. Then

$$|M_0 - s_1| > \frac{|s_1|}{2}. \tag{7.1}$$

For $(\varrho, u) \in \operatorname{supp}(\chi(s_1)) \cap \operatorname{supp}(\chi(s_2))$,

$$\varrho^\theta - u + s_2 = \varrho^\theta - u + s_1 + (s_2 - s_1) \geq s_2 - s_1 \geq M_0 - s_1 > \frac{|s_1|}{2} \tag{7.2}$$

where the first inequality follows from the fact that $(\varrho, u) \in \operatorname{supp}\chi(s_1)$. Since for $\gamma > 3$, $\lambda < 0$, it follows using the definition of χ

$$\begin{aligned} \int_{\mathbb{H}} \chi(s_1)\chi(s_2)\, d\nu_{x,t} &= \int_{\mathbb{H}} \chi(s_1)(\varrho^\theta - u + s_2)_+^\lambda (\varrho + u - s_2)_+^\lambda \, d\nu_{x,t} \\ &\leq 2^{-\lambda}|s_1|^\lambda \int_{\operatorname{supp}\chi(s_2)} \chi(s_1)(\varrho^\theta + u - s_2)_+^\lambda \, d\nu_{x,t}. \end{aligned} \tag{7.3}$$

Integrating (7.3) in s_2 over (M_0, M_0+1) yields

$$\begin{aligned} &\int_{M_0}^{M_0+1} \int_{\mathbb{H}} \chi(s_1)\chi(s_2)\, d\nu_{x,t}\, ds_2 \\ &\leq 2^{-\lambda}|s_1|^\lambda \int_{M_0}^{M_0+1} \int_{\operatorname{supp}\chi(s_2)} \chi(s_1)(\varrho^\theta + u - s_2)_+^\lambda \, d\nu_{x,t}\, ds_2 \\ &\qquad = 2^{-\lambda}|s_1|^\lambda \int_{\mathbb{H}} \left(\int_{(M_0, M_0+1)\cap(u - \varrho^\theta, u + \varrho^\theta)} (\varrho^\theta + u - s_2)_+^\lambda \, ds_2 \right) d\nu_{x,t}. \end{aligned} \tag{7.4}$$

If $\varrho^\theta + u \geq M_0 + 2$, then $\varrho^\theta + u - s_2 \geq M_0 + 2 - (M_0 + 1) = 1$, so the integral in parenthesis in (7.4) reduces to

$$\int_{(M_0, M_0+1)\cap(u - \varrho^\theta, u + \varrho^\theta)} (\varrho^\theta + u - s_2)_+^\lambda \, ds_2 \leq 1. \tag{7.5}$$

If $\varrho^\theta + u < M_0 + 2$, then

$$\begin{aligned} \int_{(M_0, M_0+1)\cap(u - \varrho^\theta, u + \varrho^\theta)} (\varrho^\theta + u - s_2)_+^\lambda \, ds_2 &\leq \int_{M_0}^{M_0+1} (\varrho^\theta + u - s_2)_+^\lambda \, ds_2 \\ &\leq \frac{1}{\lambda + 1}(\varrho^\theta + u - M_0)_+^{\lambda+1} \leq \frac{2^{\lambda+1}}{\lambda + 1} \end{aligned} \tag{7.6}$$

since $\lambda + 1 = \frac{\gamma+1}{2(\gamma-1)} > 0$. Thus, there is some constant $C(\lambda)$ depending only on λ (and thus only on γ) such that

$$\int_{M_0}^{M_0+1} \int_{\mathbb{H}} \chi(s_1)\chi(s_2)\, d\nu_{x,t}\, ds_2 \leq C(\lambda)|s_1|^\lambda \overline{\chi(s_1)}. \tag{7.7}$$

However, it is noted that if s is fixed and if $\chi(s)$ is taken as a function of (ϱ, u), that

$$\operatorname{supp}\chi(s) = \{(\varrho, u) : s \in [u - \varrho^\theta, u + \varrho^\theta]\}.$$

Since J is a connected component of A, $\chi(s) > 0$ for almost all $s \in J$ by Proposition 5.2 and the definition of J (c.f. [12]). Thus, for $\overline{\chi(s_1)}\ \overline{\chi(s_2)} \neq 0$, Lemma 6.1 yields

$$(7.8) \qquad \frac{1-\theta}{\theta}\frac{1}{s_2 - s_1}\left(\frac{\overline{u\chi(s_2)}}{\overline{\chi(s_2)}} - \frac{\overline{u\chi(s_1)}}{\overline{\chi(s_1)}}\right) = \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}\ \overline{\chi(s_2)}} - 1.$$

Following the techniques in the proof of Theorem 5 in [20], taking $s_1, s_2 \to s$ above yields

$$(7.9) \qquad \frac{1-\theta}{\theta}\frac{\partial}{\partial s}\left(\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}\right) = \frac{\overline{\chi^2(s)}}{(\overline{\chi(s)})^2} - 1 \geq 0.$$

Thus,

$$\frac{1-\theta}{\theta}\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}$$

is non-decreasing in s, which means from (7.8)

$$(7.10) \qquad \frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} \geq \overline{\chi(s_2)}$$

for $s_1 < s_2$. Combining (7.7) and (7.10) yields

$$(7.11) \qquad 0 < C(M_0, \lambda) \stackrel{\text{def}}{=} \int_{M_0}^{M_0+1} \overline{\chi(s_2)}\, ds_2 \leq C(\lambda)|s_1|^{\lambda}.$$

Since $|s_1|$ has no upper bound and $\lambda < 0$, this is a contradiction. Thus J must be bounded below. A similar argument shows J must also be bounded above. □

The rest of the argument that $\nu_{x,t}$ reduces to a delta mass is based on [20, Lemma 6]. Indeed, let $J = (s_-, s_+)$ be a connected component of A. Then the values (ϱ, u) such that $\chi(s) > 0$ on $(s_+ - \varepsilon, s_+)$ must satisfy

$$u + \varrho^\theta \geq s_+ - \varepsilon.$$

However, for these values (ϱ, u), $s_- \leq u - \varrho^\theta$, so as in [12, Section 6],

$$(7.12) \qquad \lim_{s \to s_+} \frac{\overline{u\chi(s)}}{\overline{\chi(s)}} \geq \frac{s_+ + s_-}{2}$$

and

$$(7.13) \qquad \lim_{s \to s_-} \frac{\overline{u\chi(s)}}{\overline{\chi(s)}} \leq \frac{s_+ + s_-}{2}.$$

Combined with the fact that $\frac{\overline{u\chi(s)}}{\overline{\chi(s)}}$ is non-increasing, this means that

$$(7.14) \qquad \overline{\chi(s)^2} = (\overline{\chi(s)})^2.$$

Since $\nu_{x,t}$ is a probability measure, this leads to the conclusion that

$$(7.15) \qquad \nu_{x,t} = \delta_{(\varrho(x,t), m(x,t))}$$

for $\gamma > 3$.

For the case that $\gamma = 3$, the analysis is much simpler. In that case, the commutator relation (6.8) immediately leads to

$$(7.16) \qquad \overline{\chi(s_1)\chi(s_2)} = \overline{\chi(s_1)}\ \overline{\chi(s_2)}$$

and the rest of the proof follows as for the $\gamma > 3$ case from the realization that (7.14) holds for the $\gamma = 3$ case as well. Thus, (7.15) holds for $\gamma \geq 3$.

7.2. **Reduction for Small Adiabatic Constant.** The next step is to show that in the case $\gamma \in (1, 3)$ the Young measure $\nu_{x,t}$ reduces to a delta measure. As in the case where $\gamma \geq 3$, the key point is to show that connected components of the set A as defined in the previous subsection are bounded. First, the following lemma gives conditions on the entropy kernel χ used in the argument that connected components of A are bounded.

Lemma 7.2. *For $\gamma \in (1, 3)$, $\overline{\chi(s)}$ and $\overline{\chi(s_1)\chi(s_2)}$ are continuous and weakly differentiable in their respective arguments and furthermore*

$$\frac{\partial}{\partial s}\overline{\chi(s)} \overset{def}{=} \overline{\chi'(s)} \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}; L^1(\mathbb{R}))$$

and

$$\frac{\partial}{\partial s}\overline{\chi(s_1)\chi(s_2)} \overset{def}{=} \overline{\chi'(s_1)\chi(s_2)} \in L^1_{\mathrm{loc}}((0,\infty)\times\mathbb{R}; L^1(\mathbb{R}^2)).$$

The reader is referred to [12, Lemma 7.1] for the proof of this lemma. This leads to the main proposition of this subsection.

Proposition 7.2. *Let $A \overset{def}{=} \bigcup\{(u - \varrho^\theta, \varrho^\theta + u) : (\varrho, u) \in \operatorname{supp}\nu_{x,t}\}$. Then any connected component J of A is bounded.*

Proof. The proof is in the spirit as that for [12, Proposition 7.2]. As for the large γ case, the proof is by contradiction. First, assume that J is unbounded below. Let M_0 be the supremum of J, which may be infinite. Pick s_1, s_2, and s_3 such that $s_1 < s_2 < s_3 < M_0$ and $\overline{\chi(s_1)}, \overline{\chi(s_3)} \neq 0$. Using (6.8) and (7.8)

$$(s_2 - s_1)\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} + (s_3 - s_2)\frac{\overline{\chi(s_2)\chi(s_3)}}{\overline{\chi(s_3)}} = (s_3 - s_1)\overline{\chi(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}}. \tag{7.17}$$

Differentiating (7.17) by s_2 using Lemma 7.2 and dividing by $s_3 - s_1$ gives

$$\frac{s_2 - s_1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi'(s_2)}}{\overline{\chi(s_1)}} + \frac{s_3 - s_2}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi'(s_2)}}{\overline{\chi(s_3)}} + \frac{1}{s_3 - s_1}\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} - \frac{1}{s_3 - s_1}\frac{\overline{\chi(s_3)\chi(s_2)}}{\overline{\chi(s_3)}} = \overline{\chi'(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}}. \tag{7.18}$$

Next, using the fact that $\overline{\chi(s)} \to 0$ as s goes to ∞ and to M_0 as shown in [12, Proposition 7.2, Step 2], since $\overline{\chi(s)} \geq 0$ is not identically zero, there must exist some $s_2 \in J$ such that $\overline{\chi(s_2)}, \overline{\chi'(s_2)} > 0$. Since $\overline{\chi(s)}$ is continuous in s, there is some $s_3 \in (s_2, M_0)$ such that $\overline{\chi(s_3)} > 0$.

Taking $s_1 \to -\infty$ in (7.17) yields

$$\frac{\overline{\chi(s_1)\chi(s_2)}}{\overline{\chi(s_1)}} = \frac{s_3 - s_1}{s_2 - s_1}\overline{\chi(s_2)}\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} + o(1). \tag{7.19}$$

By using (7.6) and (7.7) in [12], it can be shown that

$$\left(\overline{\chi'(s_2)} - \frac{2\lambda+1}{s_2-s_1}\overline{\chi(s_2)}\right)\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} \leq o(1). \tag{7.20}$$

However, this is a contradiction in light of the way s_2 is selected and the fact that by (6.8),

$$\frac{\overline{\chi(s_1)\chi(s_3)}}{\overline{\chi(s_1)}\;\overline{\chi(s_3)}} \geq 1 \tag{7.21}$$

for any s_1, $s_3 \in J$. Thus, J is bounded below. Similar arguments show also that J is bounded above. □

In light of the above proposition and the well-known result in [14], among other places, the following result holds

Proposition 7.3. *For $\gamma \in (1,3)$, the Young measure $\nu_{x,t}$ is a delta mass in the coordinates (ϱ, m), that is,*

$$\nu_{x,t} = \delta_{(\varrho(x,t),u(x,t))}. \tag{7.22}$$

Thus, in light of similar arguments as in [12, Section 8], this proves Theorem 1.1.

References

1. Giovanni Alberti and Stefan Müller, *A new approach to variational problems with multiple scales*, Comm. Pure Appl. Math. **54** (2001), no. 7, 761–825. MR 1823420
2. A. A. Amsden, *Kiva-3V Release 2, Improvements to Kiva-3V*, Tech. report, Los Alamos National Laboratory, 1999.
3. A. A. Amsden, P. J. O'Rourke, and T. D. Butler, *Kiva-2, a computer program for chemical reactive flows with sprays.*, Tech. report, Los Alamos National Laboratory, 1989.
4. J. M. Ball, *PDEs and continuum models of phase transitions: Proceedings of an NSF-CNRS joint seminar held in Nice, France, January 18–22, 1988*, ch. A version of the fundamental theorem for Young measures, pp. 207–215, Springer Berlin Heidelberg, Berlin, Heidelberg, 1989.
5. Joshua Ballew and Konstantina Trivisa, *Suitable weak solutions and low stratification singular limit for a fluid particle interaction model*, Quart. Appl. Math. **70** (2012), no. 3, 469–494. MR 2986131
6. ———, *Viscous and inviscid models in fluid-particle interaction*, Commun. Inf. Syst. **13** (2013), no. 1, 45–78. MR 3216986
7. ———, *Weakly dissipative solutions and weak-strong uniqueness for the Navier-Stokes-Smoluchowski system*, Nonlinear Anal. **91** (2013), 1–19. MR 3081207
8. C. Baranger, L. Boudin, P.-E. Jabin, and S. Mancini, *A modeling of biospray for the upper airways*, CEMRACS 2004—mathematics and applications to biology and medicine, ESAIM Proc., vol. 14, EDP Sci., Les Ulis, 2005, pp. 41–47 (electronic). MR 2226800
9. Stefan Berres, Raimund Bürger, Kenneth H. Karlsen, and Elmer M. Tory, *Strongly degenerate parabolic-hyperbolic systems modeling polydisperse sedimentation with compression*, SIAM J. Appl. Math. **64** (2003), no. 1, 41–80. MR 2029124
10. J. A. Carrillo, T. Karper, and K. Trivisa, *On the dynamics of a fluid-particle interaction model: the bubbling regime*, Nonlinear Anal. **74** (2011), no. 8, 2778–2801. MR 2776527 (2012d:76134)
11. José A. Carrillo and Thierry Goudon, *Stability and asymptotic analysis of a fluid-particle interaction model*, Comm. Partial Differential Equations **31** (2006), no. 7-9, 1349–1379. MR 2254618 (2007m:35193)
12. Gui-Qiang Chen and Mikhail Perepelitsa, *Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow*, Comm. Pure Appl. Math. **63** (2010), no. 11, 1469–1504. MR 2683391 (2012a:35235)

13. Gui-Qiang G. Chen and Mikhail Perepelitsa, *Vanishing viscosity solutions of the compressible Euler equations with spherical symmetry and large initial data*, Comm. Math. Phys. **338** (2015), no. 2, 771–800. MR 3351058
14. Constantine M. Dafermos, *Hyperbolic conservation laws in continuum physics*, third ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 325, Springer-Verlag, Berlin, 2010. MR 2574377 (2011i:35150)
15. J. Dolbeault, *Free energy and solutions of the Vlasov-Poisson-Fokker-Planck system: external potential and confinement (large time behavior and steady states)*, J. Math. Pures Appl. (9) **78** (1999), no. 2, 121–157. MR 1677677 (99m:35248)
16. Lawrence C. Evans, *Weak convergence methods for nonlinear partial differential equations*, CBMS Regional Conference Series in Mathematics, vol. 74, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1990. MR 1034481 (91a:35009)
17. Eduard Feireisl, *Dynamics of viscous compressible fluids*, Oxford Lecture Series in Mathematics and its Applications, vol. 26, Oxford University Press, Oxford, 2004. MR 2040667 (2005i:76092)
18. Eduard Feireisl and Antonín Novotný, *Singular limits in thermodynamics of viscous fluids*, Advances in Mathematical Fluid Mechanics, Birkhäuser Verlag, Basel, 2009. MR 2499296 (2011b:35001)
19. Philippe G. LeFloch and Michael Westdickenberg, *Finite energy solutions to the isentropic Euler equations with geometric effects*, J. Math. Pures Appl. (9) **88** (2007), no. 5, 389–429. MR 2369876
20. P.-L. Lions, B. Perthame, and E. Tadmor, *Kinetic formulation of the isentropic gas dynamics and p-systems*, Comm. Math. Phys. **163** (1994), no. 2, 415–431. MR 1284790 (95c:76093)
21. A. Mellet and A. Vasseur, *Asymptotic analysis for a Vlasov-Fokker-Planck/compressible Navier-Stokes system of equations*, Comm. Math. Phys. **281** (2008), no. 3, 573–596. MR 2415460 (2010f:35312)
22. L. Tartar, *Compensated compactness and applications to partial differential equations*, Nonlinear analysis and mechanics: Heriot-Watt Symposium, Vol. IV, Res. Notes in Math., vol. 39, Pitman, Boston, Mass.-London, 1979, pp. 136–212. MR 584398
23. F. A. Williams, *Spray combustion and atomization*, Physics of Fluids **1** (1958), no. 6, 541–545.
24. ———, *Combustion theory: The fundamental theory of chemically reacting flow systems*, 2nd ed., Benjamin/Cummings Pub., Menlo Park, CA, 1985.

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