Bose-Einstein Condensation and Global Dynamics of Solutions to a Hyperbolic Kompaneets Equation

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Abstract In this article, a simplified, hyperbolic model of the non-linear, degenerate parabolic Kompaneets equation for the number density of photons is considered. It is shown that for non-negative, compactly supported initial data, weak solutions obeying a Kružkov entropy condition are unique. Other consequences for entropy solutions resulting from a contraction estimate are explored. Certain properties of entropy solutions are investigated and convergence in time of entropy solutions with compactly supported initial data to stationary solutions is shown. The development of a Bose-Einstein condensate for initial data under certain conditions is proven. It is also shown that the total number of photons not in a Bose-Einstein condensate is non-increasing in time, and that any such loss of photons is only to the condensate.

1 Introduction

Compton scattering is the dominant process for energy transport in low-density or high-temperature plasmas. The seminal work of Kompaneets [6], which was published in 1957, derives an equation modeling the behavior of this scattering. Kompaneets' work today has applications in several areas of astrophysics including the interaction between matter and radiation early in the history of the universe and black holes [2, 9, 10].

In his work, Kompaneets considers the regime of a non-relativistic, spatially uniform, and isotropic plasma at a constant temperature and derives a Fokker-Planck approximation for the Boltzmann-Compton scattering. The photons' heat capacities are taken to be negligible. The equation that governs the evolution of the photon density f is

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$$\partial_t f = \frac{1}{x^2}\partial_x \left[x^4 \left(\partial_x f + f + f^2\right)\right]. \tag{1}$$

The photon density f is a function of the non-dimensionalized energy $x \in (0,\infty)$ and time $t \in [0,\infty)$. The total number of photons is given by

$$N(t) := \int_0^\infty x^2 f(x,t)\ \mathrm{d}x \tag{2}$$

and the total energy of the photons is given by

$$E(t) := \int_0^\infty x^3 f(x,t)\ \mathrm{d}x. \tag{3}$$

It is also known that (1) possesses an entropy structure. More specifically, the quantum entropy

$$H(t) := \int_0^\infty x^2 h(x, f(x,t))\ \mathrm{d}x \tag{4}$$

where

$$h(x,y) := y\ln y - (1+y)\ln(1+y) + xy \tag{5}$$

formally dissipates in time (see [4, 8]). This suggests that the solutions to (1) converge to some equilibrium solution as $t \to \infty$. Such non-negative solutions are given by

$$f_\mu(x) := \frac{1}{e^{x+\mu} - 1} \tag{6}$$

for $\mu \geq 0$. Taking the total photon number for each of these equilibrium solutions by using (2) yields an upper bound for the total number of photons at equilibrium of

$$\sup_{\mu \geq 0} \int_0^\infty \frac{x^2}{e^{x+\mu} - 1}\ \mathrm{d}x = 2\zeta(3) < \infty$$

(see, for example, [1]). Thus, if the initial photon number is greater than this quantity, there must be some loss of photons as $t \to \infty$. However, the Kompaneets equation for photon number density

$$\partial_t n = \partial_x \left[\left(x^2 - 2x\right)n + n^2 + x^2\partial_x n\right)\right] \tag{7}$$

shows that formally, the total photon number must be conserved, which is not possible if the initial photon number is large.

Previous work on the Kompaneets equation suggests that an out-flux of photons at $x = 0$ can occur due to a concentration of low-energy photons. In the literature, this is interpreted as a Bose-Einstein condensate. It is noted, however, that physically, there may be other effects at play, such as Bremsstrahlung radiation which would tend to suppress such an out-flux at $x = 0$. It is still worthwhile to investigate (1) mathematically to increase the understanding of how a photon flux at $x = 0$ can develop due to Compton scattering. Following conventions in previous work, the $x = 0$ out-flux is referred to as a Bose-Einstein condensate.

In order to investigate the Kompaneets equation, this work considers a hyperbolic model obtained from (7) by neglecting the diffusive term $\partial_x \left(x^2 \partial_x n\right)$:

$$\partial_t n + \partial_x \left[\left(2x - x^2\right) n - n^2\right] = 0. \tag{8}$$

The work in [8] found the omitted diffusion term to have a negligible contribution to the flux at $x = 0$ in the limit of small x. In addition, the model (8) has infinitely many stationary solutions. The largest of these agrees with the classical Bose-Einstein distribution near $x = 0$.

The model (8) is considered on the domain $x > 0$, $t > 0$. It is also assumed that as $x \to \infty$,

$$F(x,n) := \left(x^2 - 2x\right) n - n^2 \to 0$$

based on physical considerations. There is *no boundary condition imposed at zero*. Even so, there is a uniqueness result for this model. For convenience, the initial data will have compact support; this property is propagated to all times $t > 0$.

The rest of this article is a summary of the results for the dynamics of (8) from [1]. In Section 2, the notion of solution to be considered is defined and the main results stated. In Section 3, results related to the L^1-contraction for (8) are discussed. In Section 4, the regularity and compactness of solutions is investigated. Section 5 explains how the lemmas and propositions lead to the results in the main theorem and also proves some corollaries the main result. Finally, in Section 6, future research plans are discussed.

2 Definitions and Main Results

In this article, the concern is with entropy solutions (8). These are defined with the following definition.

Definition 1. Let $T > 0$. The function $n : [0,T] \times [0,\infty)$ is a *weak solution* to (8) if

$$n \in L^1\left([0,T] \times [0,\infty)\right) \cap L^1\left([0,T]; L^2[0,\infty)\right)$$

and for each test function $\phi \in C_c^\infty\left((0,T) \times (0,\infty)\right)$,

$$\int_0^T \int_0^\infty n(x,t)\partial_t \phi(x,t) + F(x,n(x,t))\partial_x \phi(x,t) \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{9}$$

In addition, n is called an *entropy solution* to (8) if for each non-negative test function $\phi \in C_c^\infty\left((0,T) \times (0,\infty)\right)$,

$$\begin{aligned}\int_0^T \int_0^\infty |n-k|\,\partial_t \phi + \operatorname{sgn}(n-k)\left[F(x,n) - F(x,k)\right]\partial_x \phi \\ -\operatorname{sgn}(n-k) F_x(x,k)\phi \,\mathrm{d}x\,\mathrm{d}t \geq 0\end{aligned} \tag{10}$$

for any $k \in \mathbb{R}$. The formulation (10) is called the *Kružkov entropy* (see [7]).

The existence of entropy solutions is proved using a vanishing viscosity technique along the lines of [7, Sections 4 and 5]. The proof of the following is found in [1, Section 3.3] and uses standard extension and vanishing viscosity techniques (see [3, 5] for example), and so is omitted here.

Proposition 1. *Let $n_0 \in L^1[0,\infty)$ be non-negative with support on $[0,R]$ for some $R > 2$. Then there exists a non-negative entropy solution to (8) in the sense of Definition 1.*

The main result of the analysis of the hyperbolic Kompaneets equation considered here and proven in the ensuing sections is as follows.

Theorem 1. *Let $n_0 \in L^1[0,\infty)$ be non-negative and compactly supported on $[0,R]$ for some $R > 2$. Then there exists a unique, non-negative, global-in-time entropy solution n to (8) such that*

$$\begin{gathered} n \in L^\infty\left([0,\infty), L^1[0,\infty)\right) \\ \left(1-e^{-t}\right) n(\cdot,t) \in L^\infty\left([0,\infty), \mathrm{BV}[0,\infty)\right) \end{gathered} \tag{11}$$

where the boundary condition $F(x,n) \to 0$ as $x \to \infty$ is satisfied in the L^1 sense. In addition, the solution satisfies the following.

1. *There exists a unique $\alpha \in [0,2]$ such that*

$$\lim_{t\to\infty} \int_0^\infty |n(x,t) - n_\alpha(x)| \, \mathrm{d}x = 0. \tag{12}$$

Here, the n_α's are the equilibrium entropy solutions defined by

$$n_\alpha(x) := \begin{cases} 0, & x \notin (\alpha,2) \\ 2x - x^2, & x \in (\alpha,2). \end{cases} \tag{13}$$

2. *The total photon number $N(t)$ is non-increasing in time. Indeed, the total photon number obeys the loss formula*

$$N(T) + \int_0^T n^2(0,t) \, \mathrm{d}t = N(0) \tag{14}$$

for $T > 0$.

3 Contraction

The key result of this section is to use the structure of entropy solutions in the sense of Definition 1 to prove the following L^1 contraction property. In light of this, it is clear that the L^1 distance between any bounded, non-negative entropy solution and any stationary solution n_α defined above is non-increasing in time.

Proposition 2 (Contraction Principle). *Let n and m be two non-negative, bounded entropy solutions of (8) in the sense of Definition 1 with L^1 initial data $n(\cdot,0)$ and $m(\cdot,0)$, respectively. Then for $R>2$,*

$$\int_0^R |n(x,T)-m(x,T)| \;\mathrm{d}x + \int_0^T \left|n(0,t)^2 - m(0,t)^2\right| \;\mathrm{d}t$$
$$\leq \int_0^R |n(x,0)-m(x,0)| \;\mathrm{d}x + \int_0^T |F(R,n(R,t)) - F(R,m(R,T))| \;\mathrm{d}t. \qquad (15)$$

It is noted that Proposition 2 is similar to the L^1 contraction result in [7]. However, Proposition 2 is different because the flux in (8) does not have the same Lipschitz property as used in [7]. This problem also has a boundary unlike the Cauchy problem.

Proposition 2 follows immediately from the following two lemmas by using $a=1$ and $b=0$ in the definition of Ψ below.

Lemma 1. *Let n and m be entropy solutions to (8) in the sense of Definition 1. Then for $\Psi(s) := a|s| + bs$ where $a \geq 0$ and $b \in \mathbb{R}$, for any non-negative test function ϕ,*

$$\int_0^T \int_0^\infty \Psi'(n(x,t)-m(x,t))\,[F(x,n)-F(x,m)]\,\partial_x\phi$$
$$+\Psi(n(x,t)-m(x,t))\partial_t\phi \;\mathrm{d}x\;\mathrm{d}t \geq 0. \qquad (16)$$

Proof. This lemma is proven by using a family of test functions

$$g_h(x,t,y,s) := \phi\left(\frac{x+y}{2},\frac{t+s}{2}\right)\eta_h\left(\frac{t-s}{2}\right)\eta_h\left(\frac{x-y}{2}\right)$$

where

$$\eta_h(x) := \frac{1}{h}\eta\left(\frac{x}{h}\right)$$

for $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta(x) \geq 0$, $\eta(x) = 0$ for $|x| \geq 1$ and

$$\int_{\mathbb{R}} \eta(x)\;\mathrm{d}x = 1.$$

Using $m(y,s)$ as k in the entropy condition for $n(x,t)$ with g_h as the test function and integrating over y and s, and doing a similar procedure for the entropy formulation of $m(y,s)$, and for the weak formulations for n and m, some elementary calculations and taking $h \to 0$ yields (16). The reader is referred to the proof of Lemma 3.4 in [1] for the details. □

Using the previous result, the following lemma immediately yields the result of Proposition 2.

Lemma 2. *Let n and m be bounded entropy solutions to (8) with initial data $n(\cdot,0)$ and $m(\cdot,0)$, respectively. Then*

$$\int_0^R \Psi(n(x,T)-m(x,T))\ \mathrm{d}x \le \int_0^R \Psi(n(x,0)-m(x,0))\ \mathrm{d}x$$
$$-\int_0^T \Psi'(n(R,t)-m(R,T))[F(R,n(R,t))-F(R,m(R,t))]\ \mathrm{d}t$$
$$+\int_0^T \Psi'(n(0,t)-m(0,t))[F(0,n(0,t))-F(0,m(0,t))]\ \mathrm{d}t. \tag{17}$$

Proof. This lemma is proven by using the test function

$$\phi(x,t) = [\alpha_h(t-\varepsilon)-\alpha_h(t-T+\varepsilon)]\,[\alpha_h(x-\varepsilon)-\alpha_h(x-R+\varepsilon)] \tag{18}$$

where

$$\alpha_h(x) := \int_{-\infty}^{x} \eta_h(s)\ \mathrm{d}s$$

in (16). This ϕ is an approximation for the characteristic function on the space-time domain $[0,T]\times[0,R]$. Taking $h\to 0$ and $\varepsilon\to 0$ completes the proof. The reader is referred to the proof of Lemma 3.5 in [1] for the details. □

The next lemma shows that for entropy solutions, if n is initially compactly supported, it remains so for all time.

Lemma 3. *Let n be a non-negative entropy solution of (8). Assume that $n(\cdot,0)$ is compactly supported on $[0,R]$ for some $R>2$. Then $n(\cdot,T)$ is compactly supported on $[0,R]$ for all $T>0$.*

Proof. Using $m\equiv 0$ and following a similar technique for proving (17), the following holds:

$$\int_R^\infty |n(x,T)|\ \mathrm{d}x - \int_0^T |n(R,t)|\left(2R-R^2-n(R,t)\right)\ \mathrm{d}t \le \int_R^\infty |n_0|\ \mathrm{d}x. \tag{19}$$

Since R is larger than 2 and n is non-negative, the result follows immediately from (19). □

Remark 1. It can be shown that in fact, the support of n contracts to $[0,2]$ as $T\to\infty$, which agrees with formal calculation of the characteristics of (8). See [1] for the details.

It is also noted that Lemma 2 also leads to the following comparison principle.

Proposition 3 (Comparison Principle). *Let n and m be non-negative entropy solutions to (8) with compactly supported initial data $n(\cdot,0)$ and $m(\cdot,0)$, respectively. If $n(x,0)\le m(x,0)$ on $(0,\infty)$, then for all $T>0$, $n(x,T)\le m(x,T)$ almost everywhere on $(0,\infty)$.*

Proof. Letting $a=b=\frac{1}{2}$ in the definition of Ψ in (16), Ψ becomes the positive part of s. Letting R be an upper bound for the supports of $n(\cdot,0)$ and $m(\cdot,0)$ and using the definitions of F and the fact that n and m are non-negative,

$$\int_0^R [n(x,T)-m(x,T)]_+ \, \mathrm{d}x \leq \int_0^R [n(x,0)-m(x,0)]_+ \, \mathrm{d}x. \tag{20}$$

The result follows immediately. □

In light of Proposition 2 and Lemma 3, it becomes clear that non-negative entropy solutions are unique.

Proposition 4 (Uniqueness of Entropy Solutions). *Let $n(\cdot,0) \in L^1(0,\infty)$ be non-negative and compactly supported on $[0,R]$. Then there is at most one non-negative entropy solution to (8) with initial data $n(\cdot,0)$.*

Proof. Let n and m be non-negative entropy solutions to (8) with initial data $n(\cdot,0)$. Using Proposition 2 along with the fact that the support for n and m is in $[0,R]$ for all positive times from Lemma 3, it is clear that

$$\int_0^R |n(x,T)-m(x,T)| \, \mathrm{d}x \leq -\int_0^T \left|n(0,t)^2 - m(0,t)^2\right| \, \mathrm{d}t. \tag{21}$$

This is only possible if $n = m$. □

4 Regularity and Compactness

In this section, the regularity and compactness of the entropy solutions are investigated. Particularly, the interest is in BV bounds for the entropy solutions as $t \to \infty$. With these bounds, the convergence to stationary solutions can be shown. All entropy solutions here are taken to be non-negative with non-negative, compactly supported initial data.

To begin with, a bound on the entropy solutions as $t \to \infty$ by stationary solutions is shown.

Lemma 4. *Let n be an entropy solution to (8). Then*

$$\limsup_{t\to\infty} n(x,t) \leq n_0 = \left(2x - x^2\right)_+. \tag{22}$$

Formally, the main idea for the proof of Lemma 4 is to find a hyperbolic counterpart to the idea of super-solution for parabolic equations, that is, to find some function $\overline{n}$ such that

$$\partial_t \overline{n} + \partial_x \overline{F} \geq 0 \tag{23}$$

where $\overline{F} := F(x,\overline{n})$.

Formally, $\overline{n}$ is chosen such that

$$\left(2x - x^2\right)\overline{n} - \overline{n}^2 = -K(t)G(x), \tag{24}$$

or

$$\overline{n}(x,t) = \frac{1}{2}\left(g + \sqrt{g^2 + 4KG}\right) \tag{25}$$

where $g(x) := 2x - x^2$. The functions K and G are chosen such that (23) holds. Some straight-forward calculations and the choice that K satisfies $\partial_t K \leq 0$ leads to the realization that

$$K(t) = \frac{1}{(\beta t + c_1)^2} \tag{26}$$

and

$$G(x) = \frac{\beta^2 (R + c_2 - x)^2}{4} \tag{27}$$

satisfy (23) for non-negative constants β, c_1 and c_2. If the hyperbolic equation (8) has a notion of entropy super-solutions and a comparison principle (note the comparison result proved in Proposition 3 only compares entropy solutions, not super-solutions), then noting that choosing $c_1 = 0$ and $c_2 > 0$ in the formulas for K and G formally would yield a super-solutions with initial values $\overline{n}(\cdot, 0) = \infty$. Then, it could easily be shown that

$$n(x,t) \leq \overline{n}(x,t) \to g(x)_+ = n_0(x) \tag{28}$$

with the limit being taken as $t \to \infty$, which would prove Lemma 4.

This argument is made rigorous by considering the viscous limit of

$$\partial_t n_\varepsilon + \partial_x \hat{F}(x, n_\varepsilon) = \varepsilon \partial_x^2 n_\varepsilon \tag{29}$$

where $\hat{F}$ is an appropriate extension of F over the entire real line. The details of this analysis are rather technical and the reader is referred to Section 4.1 in [1].

The next key step in gathering BV bounds is to prove the following one-sided spatial Lipschitz bound for entropy solutions.

Lemma 5. *Let n be an entropy solution to (8) with non-negative L^1 initial data supported on $[0,R]$ for some $R > 0$. Then for any $t > 0$, there exists some negative function $\underline{m}(R,t)$ increasing in t such that for all $0 \leq x \leq y \leq R$,*

$$n(y,t) - n(x,t) \geq \underline{m}(R,t)(y - x). \tag{30}$$

Proof. As in the proof of Lemma 4, the fact that n can be written as a vanishing viscosity limit is exploited. Letting $m_\varepsilon = \partial_x n_\varepsilon$, differentiating (29) with respect to x yields

$$\partial_t m_\varepsilon - 2m_\varepsilon^2 + 2g' m_\varepsilon + g'' n_\varepsilon + (g - 2n_\varepsilon)\partial_x m_\varepsilon - \varepsilon \partial_x^2 m_\varepsilon = 0. \tag{31}$$

Defining

$$\underline{m}_\varepsilon(t) := -\frac{C_1}{4} - \frac{\sqrt{8C_\varepsilon + C_1^2}\left(1 + \exp\left(-t\sqrt{8C_\varepsilon + C_1^2}\right)\right)}{4\left(1 - \exp\left(-t\sqrt{8C_\varepsilon + C_1^2}\right)\right)} \tag{32}$$

where

$$C_1 := 2\|g'\|_\infty \text{ and } C_\varepsilon := \sup\left(g'' n_\varepsilon\right), \tag{33}$$

straight-forward calculations show that $\underline{m}_\varepsilon$ is a sub-solution of (31) such that $\underline{m}_\varepsilon(0) = -\infty$. Thus, taking the limit as $\varepsilon \to 0$, the function $\underline{m}(R,t)$ can be defined as

$$\underline{m}(R,t) := -\frac{C_1}{4} - \frac{\sqrt{1+C_1^2}}{2\left(1-\exp\left(-t\sqrt{1+C_1^2}\right)\right)} \tag{34}$$

from which it can be shown that for any $0 \le x \le y \le R$ and $t > 0$,

$$n(y,t) - n(x,t) = \lim_{\varepsilon \to 0} n_\varepsilon(y,t) - n_\varepsilon(x,t) \ge \underline{m}(R,t)(y-x), \tag{35}$$

completing the proof. The omitted details justifying the taking of the limit $\varepsilon \to 0$ are in Section 4.2 of [1]. □

This section is concluded with a lemma specifying the compactness of the trajectory $\{n(\cdot,t)\}_{t\ge 0}$. The lemma is proven by using the L^1 contraction result of Proposition 2 and control of the total variation using the one-sided Lipschitz bound of Lemma 5 (see [1, Section 4.3]) for the details).

Lemma 6. *Let n be a non-negative entropy solution of (8) with initial data $n(\cdot,0)$ supported on $[0,R]$ for some $R > 0$. Then (11) holds and the trajectory $\{n(\cdot,t)\}_{t\ge 0}$ is relatively compact in L^1.*

5 Proof of the Main Theorem

The focus now turns to the proof of Theorem 1. The solution space for n is proven by Lemma 6.

Also from Lemma 6, the trajectory $\{n(\cdot,t)\}_{t\ge 0}$ is relatively compact in L^1. Thus, it suffices to show that subsequential limits are unique in order to show that $n(\cdot,t)$ converges in L^1 as $t \to \infty$. Let (t_k) be a sequence of times such that $t_k \to \infty$ and $n_\infty \in L^1$ such that $n(\cdot,t_k) \to n_\infty$ as $k \to \infty$. Defining for $\beta \in [0,2]$

$$C_\beta(t) := \int_0^\infty |n(x,t) - n_\beta(x)| \, \mathrm{d}x \tag{36}$$

and letting $r < t$, the results of Section 3 lead to

$$C_\beta(t) \le C_\beta(r) - \int_r^t n(0,s)^2 \, \mathrm{d}s \le C_\beta(r). \tag{37}$$

Thus, $C_\beta(t)$ is monotone and converges to some $\overline{C}_\beta$ as $t \to \infty$ which is independent of the sequence (t_k). Additionally,

$$\overline{C}_\beta = \int_0^\infty \left|n_\infty - n_\beta\right| \, \mathrm{d}x. \tag{38}$$

Since $n_\infty(x) \le n_0(x)$ by Lemma 4,

$$\overline{C}_\beta = \int_0^\beta n_\infty \, \mathrm{d}x + \int_\beta^2 n_0 - n_\infty \, \mathrm{d}x, \tag{39}$$

thus,

$$2n_\infty(\beta) = n_0(\beta) + \partial_\beta \overline{C}_\beta \tag{40}$$

and n_∞ is determined by $\overline{C}_\beta$, proving the limit does not depend on the sequence (t_k). Using standard arguments, it is clear that n_∞ is a stationary entropy solution of (8). This completes proof of (12).

The loss formula (14) follows from using the test function

$$\phi(x,t) = [\alpha_h(t-\varepsilon) - \alpha_h(t-T+\varepsilon)]\,[\alpha_h(x-\varepsilon) - \alpha_h(x-R+\varepsilon)] \tag{41}$$

in the weak formulation (9). Using similar techniques as used to prove Proposition 2, the loss formula (14) is obtained after noting that

$$\lim_{R\to\infty} \int_0^T F(R, n(R,t)) \, \mathrm{d}t = 0 \tag{42}$$

from the compact support of n. Thus, Theorem 1 is proven.

From the main result, some corollaries arise.

Corollary 1. *Let n be a non-negative entropy solution to (8) with compactly supported initial data $n(\cdot,0) \in L^1$. If*

$$\int_0^\infty n(x,0) \, \mathrm{d}x > \int_0^2 2x - x^2 \, \mathrm{d}x, \tag{43}$$

then there exists $T > 0$ such that

$$\int_0^\infty n(x,T) \, \mathrm{d}x < \int_0^\infty n(x,0) \, \mathrm{d}x \tag{44}$$

and therefore a Bose-Einstein condensate forms in finite time.

Proof. From (13), it is clear that

$$\lim_{t\to\infty} N(t) = \int_\alpha^2 2x - x^2 \, \mathrm{d}x \le \int_0^2 2x - x^2 \, \mathrm{d}x. \tag{45}$$

Thus, if

$$N(0) > \int_0^2 2x - x^2 \, \mathrm{d}x, \tag{46}$$

it must be true that for some $T > 0$,

$$N(T) < \frac{1}{2}\left(N(0) + \int_0^2 2x - x^2 \, \mathrm{d}x\right) < N(0), \tag{47}$$

implying the result. □

The next corollary states that even though it cannot be determined which stationary solution an entropy solution approaches as $t \to \infty$, a non-zero lower bound on the total photon number in the equilibrium state can be found.

Corollary 2. *Let n be a non-negative entropy solution to (8) with compactly supported, L^1 initial data $n(\cdot,0)$. Let n_α be the limiting stationary solution as $t \to \infty$. Then*

$$\int_\alpha^2 2x - x^2 \,\mathrm{d}x \geq \sup_{t\geq 0} \int_0^2 \min\{n(x,t), 2x - x^2\} \,\mathrm{d}x. \tag{48}$$

Further, if $n(\cdot,0)$ is not identically zero, then $\alpha < 2$.

Proof. Let $t_0 > 0$ and let $\underline{n}$ be the solution of (8)with initial data

$$\underline{n}(\cdot,0) = \min\left\{n(x,t_0), (2x - x^2)_+\right\}.$$

Then $\underline{n}(x,t) \leq \min\left\{n(x,t_0+t), (2x-x^2)_+\right\}$ and

$$\int_\alpha^2 2x - x^2 \,\mathrm{d}x \geq \int_0^2 \min\left\{n(x,t_0), 2x - x^2\right\} \,\mathrm{d}x \tag{49}$$

by the loss formula (14) and the fact that $\underline{n}(0,t) = 0$. This proves (48).

The fact that $\alpha < 2$ follows from the fact that entropy solutions can only have upward jumps by virtue of Lemma 5. For the details, see [1]. □

6 Future Directions

The current and future work of the authors of [1] is to find an analogous result for the full Kompaneets equation (7) to the results for (8) presented here and in [1]. Considering the full Kompaneets equation (7), it can be shown that there are stationary super-solutions of the form

$$\overline{n}(x) = m(x) + \frac{x^2}{e^2 - 1} \tag{50}$$

where m solves the ordinary differential equation

$$x^2 \frac{\mathrm{d}m}{\mathrm{d}x} + g(x)m = 0 \tag{51}$$

where

$$g(x) := x^2 - 2x + \frac{2x^2}{e^x - 1}. \tag{52}$$

Preliminary analysis shows that these supersolutions are well-behaved at $x = 0$. Thus, the plan is to use the techniques outlined in [1] and in [8] in application to the full Kompaneets equation and determine if similar results, such as showing convergence of solutions to a stationary solution and finding conditions for which it is known a Bose-Einstein condensate can form in finite time. It is anticipated that the physical entropy $H(t)$ defined in (4) will be needed to perform some of the estimates.

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