Entropy and Relative Entropy

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Outline

Hyperbolic PDEs

Entropy/Entropy Flux Pairs

Relative Entropy
- Weak-Strong Uniqueness

Weak-Strong Uniqueness for the Navier-Stokes-Smoluchowski System

Hypberolic PDEs

Consider the equation

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha(U) = 0 \tag{1}$$

where

$$x \in \mathbb{R}^m,\ t > 0, \text{ and } U : \mathbb{R}^m \times (0, \infty) \mapsto \mathbb{R}^n$$

and

$$G : \mathbb{R}^n \mapsto \mathbb{R}^m$$

Equation (1) is *hyperbolic* if and only if the matrix

$$\sum_{\alpha=1}^{m} \nu_\alpha D G_\alpha,$$

where $D := \nabla_U$ is non-singular and has n real eigenvalues and n independent eigenvectors.

For simplicity, we focus on the case $m = 1$. Then (1) becomes with some minor relabeling

$$\partial_t U + \partial_x G(U) = 0 \tag{2}$$

Examples include the inviscid Burgers' equation ($G(U) = \frac{1}{2}U^2$).

Entropy/Entropy Flux Pairs

Consider functions $\eta(U, x, t)$ and $Q(U, x, t)$ such that

$$DQ = D\eta DG$$

η is called an *entropy* and Q and *entropy flux* for (2). Together, they are called an *entropy/entropy flux pair*.
If (2) has such a pair,

$$\partial_t \eta + \partial_x Q \leq 0.$$

For smooth solutions, the above inequality becomes an equality.

Examples

1. $n = 1$: Let η be a function of U and define
$$Q(U) := \int^U \frac{\partial \eta}{\partial U}(s)\frac{\partial G}{\partial U}(s)\mathrm{d}s$$

2. Consider the elasticity equation
$$\partial_{tt} y - \partial_x w'(\partial_x y) = 0.$$
Defining $u := \partial_x y, v := \partial_t y$, we get
$$\begin{pmatrix} v \\ u \end{pmatrix}_t + \begin{pmatrix} -w'(u) \\ -v \end{pmatrix}_x = 0$$
Taking a scalar product with $(v, w'(u))^T$ yields
$$\partial_t \left(\frac{v^2}{2} + w(u)\right) - \partial_x(w'(u)v) = 0.$$
Thus, $\eta(u) := \frac{v^2}{2} + w(u)$ and $Q(u) := -w'(u)v$.

Weak Solutions

Consider the initial value problem

$$u_t + uu_x = 0, x \in \mathbb{R}, t > 0 \tag{3}$$

$$u(x,0) = \begin{cases} 0, \ x \leq 0 \\ 1, \ x > 0 \end{cases} \tag{4}$$

This problem has no classical solution. However, we can define a *weak formulation* of the problem which will have solutions

Weak Formulation

To formally find a weak formulation, we multiply the PDE by a test function and integrate by parts. Thus, the weak formulation of (3) is

$$\int_0^\infty \int_{\mathbb{R}} u\varphi_t + \frac{1}{2}u^2\varphi_x \, \mathrm{d}x \, \mathrm{d}t + \int_0^\infty \varphi(x,0) \, \mathrm{d}x = 0 \qquad (5)$$

Note that for u to satisfy this condition requires that (5) be satisfied for all test functions φ.

Non-uniqueness of Solutions and Entropy Conditions

Hyperbolic conservation laws can often have many weak solutions.
However, not all of them are physically realizable.
Thus, we impose the entropy condition to filter out non-physical solutions.

Relative Entropy

Consider (2) endowed with an entropy/entropy-flux pair (η, Q).
We define the *relative entropy* $\mathcal{H}(U|\overline{U})$ as

$$\mathcal{H}(U|\overline{U}) := \eta(U) - \eta(\overline{U}) - D\eta(\overline{U})(U - \overline{U}) \qquad (6)$$

Note that this defintion will only consider quadratic terms, but not linear terms in the entropy.
We choose this definition because if we have that $\gamma, \Gamma > 0$ and $D^2\eta$ positive definite such that

$$\gamma\mathbb{I} \leq D^2\eta \leq \Gamma\mathbb{I},$$

then there are $c_1, c_2 > 0$ such that

$$c_1|U - \overline{U}|^2 \leq \mathcal{H}(U|\overline{U}) \leq c_2|U - \overline{U}|^2.$$

Weak-Strong Uniqueness

Consider the one-dimensional problem (the multi-dimensional problem is handled in chapter 5.3 in the book of Dafermos)

$$\partial_t U + \partial_x G(U) = 0$$

with an entropy/entropy-flux pair satisfying

$$\partial_t \eta(U) + \partial_x Q(U) \leq 0$$

Then we have for a weak solution U and strong solution $\overline{U}$

$$\partial_t (U - \overline{U}) + \partial_x (G(U) - G(\overline{U})) = 0 \quad (7)$$

$$\partial_t (\eta(U) - \eta(\overline{U})) + \partial_x (Q(U) - Q(\overline{U})) \leq 0 \quad (8)$$

Multiplying (7) by $\frac{\partial \eta}{\partial U}(\overline{U})$ and some algebra yield

$$\partial_t \mathcal{H}(U|\overline{U}) + \partial_x \mathcal{Q}(U|\overline{U}) + I \leq 0 \tag{9}$$

where

$$\mathcal{Q}(U|\overline{U}) := Q(U) - Q(\overline{U}) - \frac{\partial \eta}{\partial U}(\overline{U})(G(U) - G(\overline{U}))$$
$$I := \partial_x \left(\frac{\partial \eta}{\partial U}(\overline{U}) \right) (G(U) - G(\overline{U})) + \partial_t \left(\frac{\partial \eta}{\partial U}(\overline{U}) \right) (U - \overline{U})$$

$\mathcal{Q}(U|\overline{U})$ is called the *relative entropy flux*.

Noting that

$$\partial_t \left(\frac{\partial \eta}{\partial U}(\overline{U}) \right) = \frac{\partial^2 \eta}{\partial U^2}(\overline{U}) \partial_t U = \frac{\partial^2 \eta}{\partial U^2} \left(-\frac{\partial G}{\partial U}(\overline{U}) \partial_x \overline{U} \right)$$

$$= -\partial_x \left(\frac{\partial \eta}{\partial U}(\overline{U}) \right) \frac{\partial G}{\partial U}(\overline{U}),$$

we have

$$I = \partial_x \left(\frac{\partial \eta}{\partial U}(\overline{U}) \right) \left[G(U) - G(\overline{U}) - \frac{\partial G}{\partial U}(\overline{U})(U - \overline{U}) \right] \qquad (10)$$

If we further assume that G'' is bounded, then we can obtain

$$\partial_t \mathcal{H} + \partial_x \mathcal{Q} \leq \gamma \mathcal{H}.$$

So by integrating over time and space and assuming that $U, \overline{U} \to C$ as $|x| \to \infty$, we obtain

$$\int_{\mathbb{R}} \mathcal{H}(\tau)\ \mathrm{d}x \leq \int_{\mathbb{R}} \mathcal{H}(0)\ \mathrm{d}x + \gamma \int_0^\tau \int_{\mathbb{R}} \mathcal{H}(t)\ \mathrm{d}x\ \mathrm{d}t.$$

Using Gronwall's inequality and the fact that $|U - \overline{U}|^2$ is bounded by $C\mathcal{H}(U|\overline{U})$, if the initial data for U and $\overline{U}$ are equal, the equality continues for any time t.

Remark: This result can be generalized to multiple dimensions with similar analysis to the following theorem.

Theorem:

Assume the system

$$\partial_t U + \sum_{\alpha=1}^{m} \partial_\alpha G_\alpha U = 0 \tag{11}$$

has an entropy η such that $D^2\eta(U)$ is positive definite, uniformly on compact subsets. Suppose $\overline{U}$ is a classical solution on $[0, T)$ of (11). Let U be any suitable weak solution of (11) with the same initial data as $\overline{U}$. Then on compact subsets, the classical solution is the unique suitable weak solution of (11).

Navier-Stokes-Smoluchowski System

$$
\begin{aligned}
&\partial_t \rho + \mathrm{div}_x (\rho \mathbf{u}) = 0 \\
&\partial_t (\rho \mathbf{u}) + \mathrm{div}_x (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla_x (p(\rho) + \eta) - \mu \Delta_x \mathbf{u} - \lambda \nabla_x \mathrm{div}_x \mathbf{u} \\
&= - (\eta + \beta \rho) \nabla_x \Phi \\
&\partial_t \eta + \mathrm{div}_x (\eta (\mathbf{u} - \nabla_x \Phi)) - \Delta_x \eta = 0
\end{aligned}
$$

where $p(\rho) = a\rho^\gamma$ for some $a > 0, \gamma > 1, \beta \neq 0$

Since we also have the energy inequality

$$\mathcal{F}(\varrho, \mathbf{u}, \eta)(\tau) := \int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \log \eta + (\beta\varrho + \eta)\Phi \, \mathrm{d}x(\tau) \\ \leq \mathcal{F}(\varrho, \mathbf{u}, \eta)(0),$$

we can define the entropy $\mathcal{E}(U) := \mathcal{F}$ where

$$U = \begin{pmatrix} \varrho \\ \mathbf{m} := \varrho\mathbf{u} \\ \eta \end{pmatrix}$$

We then use the formula

$$\mathcal{H}(U|\overline{U}) := \mathcal{E}(U) - \mathcal{E}(\overline{U}) - D\mathcal{E}(\overline{U}) \cdot (U - \overline{U})$$

to obtain the relative entropy.

Weak-Strong Uniqueness for the NSS System

The relative entropy is

$$\mathcal{H}(U|\overline{U}) = \frac{\varrho}{2}|\mathbf{u}-\mathbf{U}|^2 + \frac{a}{\gamma-1}\left(\varrho^\gamma - r^\gamma\right) - \frac{a\gamma}{\gamma-1} r^{\gamma-1}(\varrho - r)$$

$$+\eta\ln\eta - s\ln s - (\ln s + 1)(\eta - s) =: \frac{\varrho}{2}|\mathbf{u}-\mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s),$$

and obeys the inequality

$$\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s)\ \mathrm{d}x(\tau)$$

$$+\int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\ \mathrm{d}t$$

$$\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0 - \mathbf{U}_0|^2 + E_F(\varrho_0, r_0) + E_P(\eta_0, s_0)\ \mathrm{d}x + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s)\ \mathrm{d}t$$

The remainder term is defined as

$$
\begin{aligned}
&\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s) \\
&:= \int_\Omega \mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{U})) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x - \int_\Omega \varrho(\partial_t \mathbf{U} + \mathbf{u} \cdot \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x \\
&- \int_\Omega \partial_t P_F(r)(\varrho - r) + \nabla_x P_F(r) \cdot (\varrho \mathbf{u} - r\mathbf{U}) \ \mathrm{d}x \\
&- \int_\Omega \left[\varrho(P_F(\varrho) - P_F(r)) - E_F(\varrho, r)\right] \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \\
&- \int_\Omega \partial_t P_P(s)(\eta - s) + \nabla_x P_P(s) \cdot (\eta \mathbf{u} - s\mathbf{U}) \ \mathrm{d}x \\
&- \int_\Omega \left[\eta(P_P(\eta) - P_P(s)) - E_P(\eta, s)\right] \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \\
&- \int_\Omega \nabla_x (P_P(\eta) - P_P(s)) \cdot (\nabla_x \eta + \eta \nabla_x \Phi) \ \mathrm{d}x \\
&- \int_\Omega (\beta\varrho + \eta) \nabla_x \Phi \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x - \int_\Omega \frac{\eta \nabla_x s}{s} \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x.
\end{aligned}
$$

Sketch of Proof

1. Devise an approximate system using vanishing viscosity and artificial pressure terms. These approximate solutions will obey an approximate system of PDEs and an approximate relative entropy inequality.
2. Show that the sequence of approximate solutions converges appropriately to solutions to the original NSS system. Also show that these solutions to the NSS system obey the relative entropy inequality given previously. This establishes the existence of the *suitable weak solutions*.
3. Use the relative entropy inequality to show the weak-strong uniqueness result.

Approximate System

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = \varepsilon \Delta_x \varrho_n \tag{12}$$

$$\partial_t \eta_n + \mathrm{div}_x(\eta_n \mathbf{u}_n - \eta_n \nabla_x \Phi) = \Delta_x \eta_n \tag{13}$$

$$\begin{aligned}
&\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{w} \, \mathrm{d}x \\
&= \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x \mathbf{w} + (a\varrho_n^\gamma + \eta_n + \delta \varrho_n^\alpha) \mathrm{div}_x \mathbf{w} \, \mathrm{d}x \\
&- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{w} + \varepsilon \nabla_x \varrho_n \cdot \nabla_x \mathbf{u}_n \cdot \mathbf{w} \, \mathrm{d}x \\
&- \int_\Omega (\beta \varrho_n + \eta_n) \nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x
\end{aligned} \tag{14}$$

NSS Weak-Strong Uniqueness Theorem

Theorem: Let $\{\varrho, \mathbf{u}, \eta\}$ be a suitable weak solution of the NSS system. Assume that $\{r, \mathbf{U}, s\}$ is a smooth solution of the NSS system with the regularity

$$
\left\{
\begin{array}{l}
r \in C_{\text{weak}}([0,T]; L^{\gamma}(\Omega)) \\
\mathbf{U} \in C_{\text{weak}}([0,T]; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \\
\nabla_x \mathbf{U} \in L^2(0,T; L^2(\Omega;\mathbb{R}^{3\times 3})), \mathbf{U}|_{\partial\Omega} = 0 \\
s \in C_{\text{weak}}([0,T]; L^1(\Omega)) \cap L^1(0,T; L^{6\gamma/\gamma-3}(\Omega)) \\
\partial_t \mathbf{U} \in L^1(0,T; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3)) \\
\nabla_x^2 \mathbf{U} \in L^1(0,T; L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^{3\times3\times3}) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^{3\times3\times3})) \\
\partial_t P_F(r) \in L^1(0,T; L^{\gamma/\gamma-1}(\Omega)) \\
\nabla_x P_F(r) \in L^1(0,T; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3)) \\
\partial_t P_P(s) \in L^1(0,T; L^{\infty}(\Omega)) \cap L^{\infty}(0,T; L^{3/2}(\Omega)) \\
\nabla_x P_P(s) \in L^{\infty}(0,T; L^3(\Omega;\mathbb{R}^3)) \\
\nabla_x s \in L^{\infty}(0,T; L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma+3}(\Omega;\mathbb{R}^3)).
\end{array}
\right.
\tag{15}
$$

In addition, assume $\{r, \mathbf{U}, s\}$ obey the hypotheses

$$\nabla_x r \in L^2(0, T; L^q(\Omega; \mathbb{R}^3)) \tag{16}$$

$$\nabla_x^2 \mathbf{U} \in L^2(0, T; L^q(\Omega; \mathbb{R}^{3\times3\times3})) \tag{17}$$

$$\omega := \nabla_x s + s\nabla_x \Phi \in L^2(0, T; L^q(\Omega; \mathbb{R}^3)) \tag{18}$$

where

$$q > \max\left\{3, \frac{3}{\gamma - 1}\right\}.$$

Then $\{\varrho, \mathbf{u}, \eta\}$ is identically $\{r, \mathbf{U}, s\}$.

References

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