# Relative Entropy and the Navier-Stokes-Smoluchowski System for Compressible Fluids

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# Outline

# Entropy/Entropy Flux Pairs

Consider the hyperbolic equation

$$\partial_t U + \operatorname{div}_x G(U) = 0 \tag{1}$$

where $U : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}^m$. Examples include the inviscid Burgers' equation ($G(U) = \frac{1}{2}U^2$). Consider functions $\mathcal{E}(U, x, t)$ and $Q(U, x, t)$ such that

$$DQ = (DG)(D\mathcal{E})$$

$\mathcal{E}$ is called an *entropy* and $Q$ and *entropy flux* for (1). Together, they are called an *entropy/entropy flux pair*.
For weak solutions to the hyperbolic problem (1),

$$\partial_t \mathcal{E} + \operatorname{div}_x Q \leq 0.$$

# Examples of Entropy/Entropy Flux Pairs I

Consider

$$\partial_t U + \partial_x G(U) = 0. \tag{2}$$

If $\mathcal{E}$ is any function, and

$$Q(U) = \int^U \mathcal{E}'(U) G'(U) \, \mathrm{d}U,$$

then $\mathcal{E}$ and $Q$ are an entropy/entropy flux pair.

# Examples of Entropy/Entropy Flux Pairs II

Consider the one-dimensional Euler equations for compressible fluids:

$$\partial_t \varrho + \partial_x(\varrho u) = 0 \tag{3}$$

$$\partial_t(\varrho u) + \partial_x \left(\varrho |u|^2 + p_F(\varrho)\right) = 0 \tag{4}$$

with $p_F(\varrho) = a\varrho^\gamma$.

We have the entropy/entropy flux pair

$$\mathcal{E}(\varrho, u) := \frac{1}{2}\varrho|u|^2 + \frac{a}{\gamma - 1}\varrho^\gamma$$

and

$$Q(\varrho, u) = \frac{1}{2}\varrho|u|^3 + \frac{a\gamma}{\gamma - 1}\varrho^\gamma u.$$

If $\psi$ is a continuous function, we also have the entropy/entropy flux pair

$$\mathcal{E}^\psi(\varrho, u) := \varrho \int_{-1}^{1} \psi(u + \varrho^{\gamma-1}s)(1 - s^2)^\lambda \, \mathrm{d}s$$

$$Q^\psi(\varrho, u) := \varrho \int_{-1}^{1} \left(u + \frac{\gamma - 1}{2}\varrho^{\gamma-1}s\right) \psi(u + \varrho^{\gamma-1}s)(1 - s^2)^\lambda \, \mathrm{d}s$$

where $\lambda = \frac{3-\gamma}{2(\gamma-1)}$.

# Relative Entropy

Consider an entropy/entropy flux pair $(\mathcal{E}, Q)$. We define the *relative entropy* $\mathcal{H}(U|\overline{U})$ as

$$\mathcal{H}(U|\overline{U}) := \mathcal{E}(U) - \mathcal{E}(\overline{U}) - D\mathcal{E}(\overline{U}) \cdot (U - \overline{U}) \tag{5}$$

Note that this definition will only consider quadratic terms, but not linear terms in the entropy.

# Example of a Relative Entropy

Consider the Euler model (3)-(4). The relative mechanical entropy is

$$\mathcal{H}(\varrho, u|\overline{\varrho}, \overline{u}) = \frac{1}{2}\varrho|u - \overline{u}|^2 + E_F(\varrho, \overline{\varrho}) \tag{6}$$

where $E_F(\varrho, \overline{\varrho}) := \frac{a}{\gamma-1}(\varrho^\gamma - \overline{\varrho}^\gamma) - \frac{a\gamma}{\gamma-1}\overline{\varrho}^{\gamma-1}(\varrho - \overline{\varrho})$.

# Fluid-Particle Interaction

- Fluid-particle interaction models are of interest to engineers and scientists studying biotechnolgy, medicine, waste-water recycling, mineral processing, and combustion theory.
- The macroscopic model considered in this talk, the Navier-Stokes-Smoluchowski system, is formally derived from a Fokker-Planck type kinetic equation coupled with fluid equations.
- This coupling is from the mutual frictional forces between the particles and the fluid, assumed to follow Stokes' Law.
- The fluid is a viscous, Newtonian, compressible fluid.

# Navier-Stokes-Smoluchowski System

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{7}$$

$$\begin{aligned} &\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \left( a\varrho^\gamma + \eta \right) \\ &= \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) - (\beta \varrho + \eta) \nabla_x \Phi \end{aligned} \tag{8}$$

$$\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u} - \eta \nabla_x \Phi) = \Delta_x \eta \tag{9}$$

$$\begin{aligned} &\int_\Omega \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{a}{\gamma - 1} \varrho^\gamma + \eta \ln \eta + (\beta \varrho + \eta)\Phi \ \mathrm{d}x(\tau) \\ &+ \int_0^\tau \int_\Omega \mu |\nabla_x \mathbf{u}|^2 + \lambda |\, \mathrm{div}_x \mathbf{u}|^2 + |\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \ \mathrm{d}x \ \mathrm{d}t \\ &\leq \int_\Omega \frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + \frac{a}{\gamma - 1} \varrho_0^\gamma + \eta_0 \ln \eta_0 + (\beta \varrho_0 + \eta_0) \nabla_x \Phi \ \mathrm{d}x \end{aligned} \tag{10}$$

# Constitutive Relations and Boundary and Initial Conditions

Newtonian Condition for a Viscous Fluid:

$$\mathbb{S}(\nabla_x \mathbf{u}) := \mu(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) + \lambda \, \mathrm{div}_x \, \mathbf{u}\mathbb{I}$$

$$\mu > 0, \ \lambda + \frac{2}{3}\mu \geq 0$$

Pressure Conditions:

$$\gamma > \frac{3}{2}, \ a > 0$$

Boundary and Initial Conditions:

$$\mathbf{u}|_{\partial\Omega} = (\nabla_x \eta + \eta \nabla_x \Phi) \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{11}$$

$$\varrho_0 \in L^\gamma(\Omega) \cap L^1_+(\Omega) \tag{12}$$

$$\mathbf{m}_0 \in L^{\frac{6}{5}}(\Omega; \mathbb{R}^3) \cap L^1(\Omega; \mathbb{R}^3) \tag{13}$$

$$\eta_0 \in L^2(\Omega) \cap L^1_+(\Omega) \tag{14}$$

# Confinement Hypotheses

Take $\Phi : \Omega \mapsto \mathbb{R}^+$ where $\Omega$ is $C^{2,\nu}$.

Bounded Domain

- $\Phi$ is bounded and Lipschitz on $\overline{\Omega}$.
- $\beta \neq 0$.
- The sub-level sets $[\Phi < k]$ are connected in $\Omega$ for all $k > 0$.

Unbounded Domain

- $\Phi \in W^{1,\infty}_{loc}(\Omega)$.
- $\beta > 0$.
- The sub-level sets $[\Phi < k]$ are connected in $\Omega$ for all $k > 0$.
- $e^{-\Phi/2} \in L^1(\Omega)$.
- $|\Delta_x \Phi(x)| \leq c_1 |\nabla_x \Phi(x)| \leq c_2 \Phi(x)$ for $x$ with sufficiently large magnitude.

# Weak Formulation I

Carrillo *et al.* (2010) established the existence of renormalized weak solutions in the following sense:
Assume that $\Phi, \Omega$ satisfy the confinement hypotheses. Then $\{\varrho, \mathbf{u}, \eta\}$ represent a *renormalized weak solution* to (7)-(10) if and only if

- $\varrho \geq 0$ in $L^\infty(0, T; L^\gamma(\Omega))$ represents a renormalized solution of (7) on $(0, \infty) \times \Omega$, i.e., for any test function $\phi \in \mathcal{D}([0, T) \times \overline{\Omega})$, $T > 0$ and any $b, B$ such that

$$b \in L^\infty([0, \infty)) \cap C([0, \infty)),\ B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2} dz,$$

  the renormalized continuity equation

$$\int_0^T \int_\Omega B(\varrho)\partial_t\phi + B(\varrho)\mathbf{u}\cdot\nabla_x\phi - b(\varrho)\phi \,\mathrm{div}_x\, \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t = -\int_\Omega B(\varrho_0)\phi(0, \cdot)\ \mathrm{d}x \tag{15}$$

  holds.

# Weak Formulation II

- The balance of momentum holds in the sense of distributions, i.e., for any $\mathbf{w} \in \mathcal{D}([0,T);\mathcal{D}(\overline{\Omega};\mathbb{R}^3))$,

$$\int_0^T \int_\Omega \varrho\mathbf{u} \cdot \partial_t \mathbf{w} + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{w} + (p_F(\varrho) + \eta) \operatorname{div}_x \mathbf{w} \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^\infty \int_\Omega \mu \nabla_x \mathbf{u} \nabla_x \mathbf{w} + \lambda \operatorname{div}_x \mathbf{u} \operatorname{div}_x \mathbf{w} - (\beta\varrho + \eta)\nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x \, \mathrm{d}t$$
$$- \int_\Omega \mathbf{m}_0 \cdot \mathbf{w}(0,\cdot) \, \mathrm{d}x \tag{16}$$

  All quantities are required to be integrable, so in particular, $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, thus the velocity field can be required to vanish on $\partial\Omega$ in the sense of traces.

- $\eta \geq 0$ is a weak solution of (9), i.e.,

$$\int_0^T \int_\Omega \eta \partial_t \phi + \eta \mathbf{u} \cdot \nabla_x \phi - \eta \nabla_x \Phi \cdot \nabla_x \phi - \nabla_x \eta \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega \eta_0 \phi(0,\cdot) \, \mathrm{d}x \tag{17}$$

  Again, terms in this equation must be integrable on $(0,T) \times \Omega$, so in particular $\eta \in L^2(0,T;L^3(\Omega)) \cap L^1(0,T;W^{1,\frac{3}{2}}(\Omega))$.

# Weak Existence

- The existence result of Carrillo *et al*. is established by implementing a time-discretization approximation supplemented with an artificial pressure approximation.
- Their paper also handles the case of unbounded domains and proves the convergence to a steady-state solution as $t \to \infty$.

# Weakly Dissipative Solutions I

Next, we define a new version of solution.

## Definition (Weakly Dissipative Solutions)

$\{\varrho, \mathbf{u}, \eta\}$ are called a *weakly dissipative solution* to the NSS system if and only if

- $\{\varrho, \mathbf{u}, \eta\}$ form a renormalized weak solution with the energy inequality

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \eta\ln\eta + \eta\Phi \; \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + |2\nabla_x\sqrt{\eta} + \sqrt{\eta}\nabla_x\Phi|^2 \; \mathrm{d}x \; \mathrm{d}t \\
&\le \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \eta_0\ln\eta_0 + \eta_0\Phi \; \mathrm{d}x \\
&- \beta\int_0^\tau \int_\Omega \varrho\mathbf{u}\cdot\nabla_x\Phi \; \mathrm{d}x \; \mathrm{d}t
\end{aligned} \tag{18}$$

  for all $\tau$.

# Weakly Dissipative Solutions II

## Definition (Weakly Dissipative Solutions)

- for all suitably smooth solutions $\{r, \mathbf{U}, s\}$ of the NSS system, the following relative entropy inequality holds for all $\tau$.

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s)\ \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\ \mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{U}_0|^2 + E_F(\varrho_0, r_0) + E_P(\eta_0, s_0)\ \mathrm{d}x \\
&+ \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s)\ \mathrm{d}t
\end{aligned} \tag{19}
$$

# Remainder Term

The remainder term in (19) has the form

$$
\begin{aligned}
&\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s) \\
&:= \int_\Omega \mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{U})) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x - \int_\Omega \varrho(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x \\
&- \int_\Omega \partial_t P_F(r)(\varrho - r) + \nabla_x P_F(r) \cdot (\varrho \mathbf{u} - r\mathbf{U}) \, \mathrm{d}x \\
&- \int_\Omega [\varrho(P_F(\varrho) - P_F(r)) - E_F(\varrho, r)] \, \mathrm{div}_x \mathbf{U} \, \mathrm{d}x \\
&- \int_\Omega \partial_t P_P(s)(\eta - s) + \nabla_x P_P(s) \cdot (\eta \mathbf{u} - s\mathbf{U}) \, \mathrm{d}x \\
&- \int_\Omega [\eta(P_P(\eta) - P_P(s)) - E_P(\eta, s)] \, \mathrm{div}_x \mathbf{U} \, \mathrm{d}x \\
&- \int_\Omega \nabla_x(P_P(\eta) - P_P(s)) \cdot (\nabla_x \eta + \eta \nabla_x \Phi) \, \mathrm{d}x \\
&- \int_\Omega \left[ (\beta\varrho + \eta)\nabla_x \Phi + \frac{\eta \nabla_x s}{s} \right] \cdot (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x
\end{aligned}
$$

Here we define

$$\begin{aligned}
H_F(\varrho) &:= \frac{a}{\gamma - 1}\varrho^{\gamma} \\
P_F(\varrho) &:= H_F'(\varrho) = \frac{a\gamma}{\gamma - 1}\varrho^{\gamma - 1} \\
E_F(\varrho, r) &:= H_F(\varrho) - H_F'(r)(\varrho - r) - H_F(r) \\
H_P(\eta) &:= \eta \ln \eta \\
P_P(\eta) &:= H_P'(\eta) = \ln \eta + 1 \\
E_P(\eta, s) &:= H_P(\eta) - H_P'(s)(\eta - s) - H_P(s)
\end{aligned}$$

# Approximation Scheme

A three-level approximation scheme is employed

- Artificial pressure parameterized by small $\delta$
- Vanishing viscosity parameterized by small $\varepsilon$
- Faedo-Galerkin approximation where test functions for the momentum equation are taken from $n$-dimensional function spaces $X_n$ of smooth functions on $\overline{\Omega}$

# Approximate System

$$\partial_t \varrho_n + \operatorname{div}_x(\varrho_n \mathbf{u}_n) = \varepsilon \Delta_x \varrho_n \tag{20}$$

$$\partial_t \eta_n + \operatorname{div}_x(\eta_n \mathbf{u}_n - \eta_n \nabla_x \Phi) = \Delta_x \eta_n \tag{21}$$

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{w} \,\mathrm{d}x = \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x \mathbf{w} + (a\varrho_n^\gamma + \eta_n + \delta \varrho_n^\alpha) \operatorname{div}_x \mathbf{w} \,\mathrm{d}x$$
$$- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{w} + \varepsilon \nabla_x \varrho_n \cdot \nabla_x \mathbf{u}_n \cdot \mathbf{w} \,\mathrm{d}x - \int_\Omega (\beta \varrho_n + \eta_n) \nabla_x \Phi \cdot \mathbf{w} \,\mathrm{d}x \tag{22}$$

with the additional conditions

$$\nabla_x \varrho_n \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial\Omega$$
$$\mathbf{u}_n = (\nabla_x \eta_n + \eta_n \nabla_x \Phi) \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial\Omega$$

# Existence of Approximate Solutions

- Existence of $\mathbf{u}_n$ is obtained from the Faedo-Galerkin approximation and an iteration argument.
- $\varrho_n, \eta_n$ obtained from $\mathbf{u}_n$ using fixed point arguments in the spirit of Ladyzhenskaya.

# Approximate Energy Inequality

Using $\mathbf{u}_n$ as a test function in (22) and some straight-forward manipulations:

$$\int_\Omega \frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \frac{a}{\gamma-1}\varrho_n^\gamma + \frac{\delta}{\alpha-1}\varrho_n^\alpha + \eta_n \ln \eta_n + \eta_n \Phi \, \mathrm{d}x(\tau)$$
$$+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n + |2\nabla_x \sqrt{\eta_n} + \sqrt{\eta_n}\nabla_x \Phi|^2 \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_n|^2 (a\gamma \varrho_n^{\gamma-2} + \delta a \varrho_n^{\alpha-2}) \, \mathrm{d}x \, \mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta}\ln \eta_{0,\delta} + \eta_{0,\delta}\Phi \, \mathrm{d}x$$
$$- \beta \int_0^\tau \int_\Omega \varrho_n \mathbf{u}_n \cdot \nabla_x \Phi \, \mathrm{d}x \, \mathrm{d}t \qquad (23)$$

# Uniform Bounds

From the energy inequality, we find that

$$
\begin{aligned}
&\{\mathbf{u}\}_{n,\varepsilon,\delta} \in_b L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))\\
&\{\sqrt{\varrho}\mathbf{u}\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T; L^2(\Omega;\mathbb{R}^3))\\
&\{\varrho\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T; L^\gamma(\Omega))\\
&\{\eta \ln \eta\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T; L^1(\Omega))\\
&\{\nabla_x\sqrt{\eta}\}_{n,\varepsilon,\delta} \in_b L^2(0,T; L^2(\Omega;\mathbb{R}^3))\\
&\{\eta\}_{n,\varepsilon,\delta} \in_b L^2(0,T; W^{1,\frac{3}{2}}(\Omega))
\end{aligned}
$$

# Faedo-Galerkin Limit I

- From the approximate energy balance, the term

$$\varepsilon\delta \int_0^T \int_\Omega |\nabla_x \varrho_n|^2 \varrho_n^{\alpha-2} \, \mathrm{d}x \, \mathrm{d}t$$

is bounded independently of $n$. Thus by Poincaré's inequality,

$$\{\varrho\}_n \in_b L^2(0, T; W^{1,2}(\Omega)).$$

- From this, $\nabla_x \varrho_n \cdot \mathbf{u}_n \in_b L^1(0, T; L^{3/2}(\Omega))$. To get higher time integrability, multiply (20) by $G'(\varrho_n)$ where $G(\varrho_n) := \varrho_n \ln \varrho_n$. Then

$$\varepsilon \int_0^T \int_\Omega \frac{|\nabla_x \varrho_n|^2}{\varrho_n} \, \mathrm{d}x \, \mathrm{d}t$$

is bounded independently of $n$. Using Hölder's and interpolation,

$$\{\nabla_x \varrho_n \cdot \mathbf{u}_n\}_n \in_b L^q(0, T; L^p(\Omega))$$

for some $p \in \left(1, \frac{3}{2}\right)$ and $q \in (1, 2)$. Thus, $\varrho_\varepsilon, \mathbf{u}_\varepsilon$ obey

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \Delta_x \varrho_\varepsilon.$$

# Faedo-Galerkin Limit II

- Strong convergence of $\nabla_x \varrho_n \to \nabla_x \varrho_\varepsilon$ follows from letting $G(z) = z^2$.
- Similar techniques show convergence of $\eta_n \to \eta_\varepsilon$ and $\nabla_x \eta_n \to \nabla_x \eta_\varepsilon$ to allow

$$\partial_t \eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon - \eta_\varepsilon \nabla_x \Phi) = \Delta_x \eta_\varepsilon.$$

- Terms in the momentum equation converge as we want using the bounds and the above convergences, except for the convective term.
- Convergence of the convective term $\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n$ in $L^q((0,T) \times \Omega; \mathbb{R}^3)$ follows from convergence of $\varrho_n \mathbf{u}_n$ and Arzela-Ascoli.

The following lemma is of use throughout the analysis for convergence of the $\eta$ terms:

## Lemma (Simon)

*Let $X \subset B \subset Y$ be Banach spaces with $X \subset B$ compactly. Then, for $1 \leq p < \infty$, $\{v : v \in L^p(0,T;X),\ v_t \in L^1(0,T;Y)\}$ is compactly embedded in $L^p(0,T;B)$.*

Thus, $\{\eta\}_{n,\varepsilon} \to \eta_\delta$ in $L^2(0,T;L^3(\Omega))$.

# Vanishing Viscosity Approximation I

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \Delta_x \varrho_\varepsilon \tag{24}$$

$$\partial_t \eta_\varepsilon + \operatorname{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon - \eta_\varepsilon \nabla_x \Phi) = \Delta_x \eta_\varepsilon \tag{25}$$

$$\int_\Omega \partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) \cdot \mathbf{w} \, \mathrm{d}x = \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w} + (a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta \varrho_\varepsilon^\alpha) \operatorname{div}_x \mathbf{w} \, \mathrm{d}x$$
$$- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{w} + \varepsilon \nabla_x \varrho_\varepsilon \cdot \nabla_x \mathbf{u}_\varepsilon \cdot \mathbf{w} \, \mathrm{d}x - \int_\Omega (\beta \varrho_\varepsilon + \eta_\varepsilon) \nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x \tag{26}$$

$$\nabla_x \varrho_\varepsilon \cdot \mathbf{n} = 0$$

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = (\nabla_x \eta_\varepsilon + \eta_\varepsilon \nabla_x \Phi) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

# Vanishing Viscosity Approximation II

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{\delta}{\alpha-1}\varrho_\varepsilon^\alpha + \eta_\varepsilon \ln \eta_\varepsilon + \eta_\varepsilon \Phi \, \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon + |2\nabla_x \sqrt{\eta_\varepsilon} + \sqrt{\eta_\varepsilon}\nabla_x \Phi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&+ \varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_\varepsilon|^2 (a\gamma \varrho_\varepsilon^{\gamma-2} + \delta a \varrho_\varepsilon^{\alpha-2}) \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta}\ln \eta_{0,\delta} + \eta_{0,\delta}\Phi \, \mathrm{d}x \\
&- \beta \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Phi \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
\tag{27}
$$

# Vanishing Viscosity Limit I

- We begin by using the uniform bounds and obtaining weak limits $\varrho_\delta, \mathbf{u}_\delta, \eta_\delta$.
- We show that since $\sqrt{\varepsilon}\nabla_x \varrho_\varepsilon \to 0$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$, and using Arzela-Ascoli, we have that $\varrho_\delta, \mathbf{u}_\delta$ solve the continuity equation weakly.
- Similar analysis shows that $\eta_\delta, \mathbf{u}_\delta$ solve the Smoluchowski equation weakly.

# Vanishing Viscosity Limit II

- Convergence of the momentum equation is fairly straight-forward except for the pressure-related terms. Using the Bogovskii operator (analogous to an inverse divergence operator) and an appropriate test function, we find that $a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha$ has a weak limit.
- To show this weak limit is $a\varrho_\delta^\gamma + \eta_\delta + \delta\varrho_\delta^\alpha$, we have to show the strong convergence of the fluid density (strong convergence of the particle density follows from the lemma of Simon).
- This is obtained by using the test function $\psi(t)\zeta(x)\varphi_1(x)$ where $\psi \in C_c^\infty(0,T), \zeta \in C_c^\infty(\Omega), \varphi_1(x) := \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon)$, and analysis involving the double Reisz transform and the Div-Curl Lemma.

# Artificial Pressure Approximation I

$$\int_0^T \int_\Omega \varrho_\delta B(\varrho_\delta)(\partial_t \phi + \mathbf{u}_\delta \cdot \nabla_x \phi) \ \mathrm{d}x \ \mathrm{d}t + \int_\Omega \varrho_{0,\delta} B(\varrho_{0,\delta}) \phi(0,\cdot) \ \mathrm{d}x$$
$$= \int_0^T \int_\Omega b(\varrho_\delta) \, \mathrm{div}_x \, \mathbf{u}_\delta \phi \ \mathrm{d}x \ \mathrm{d}t \tag{28}$$

$$\int_0^T \int_\Omega \eta_\delta \partial_t \phi + (\eta_\delta \mathbf{u}_\delta - \eta_\delta \nabla_x \Phi - \nabla_x \eta_\delta) \cdot \nabla_x \phi \ \mathrm{d}x \ \mathrm{d}t = - \int_\Omega \eta_{0,\delta} \phi(0,\cdot) \ \mathrm{d}x \tag{29}$$

$$\int_\Omega \partial_t (\varrho_\delta \mathbf{u}_\delta) \mathbf{w} \ \mathrm{d}x = \int_\Omega \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla_x \mathbf{w} + (a \varrho_\delta^\gamma + \eta_\delta + \delta \varrho_\delta^\alpha) \, \mathrm{div}_x \, \mathbf{w} \ \mathrm{d}x$$
$$- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{w} \ \mathrm{d}x - \int_\Omega (\beta \varrho_\delta + \eta_\delta) \nabla_x \Phi \cdot \mathbf{w} \ \mathrm{d}x \tag{30}$$

# Artificial Pressure Approximation II

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_\delta |\mathbf{u}_\delta|^2 + \frac{a}{\gamma-1}\varrho_\delta^\gamma + \frac{\delta}{\alpha-1}\varrho_\delta^\alpha + \eta_\delta \ln \eta_\delta + \eta_\delta \Phi \ \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{u}_\delta + |2\nabla_x \sqrt{\eta_\delta} + \sqrt{\eta_\delta}\nabla_x \Phi|^2 \ \mathrm{d}x \ \mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_{0,\delta} |\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta} \ln \eta_{0,\delta} + \eta_{0,\delta}\Phi \ \mathrm{d}x \\
&- \beta \int_0^\tau \int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \nabla_x \Phi \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}
\tag{31}
$$

# Artificial Pressure Limit

- Again, from uniform bounds, we are able to obtain the existence of weak limits $\varrho, \mathbf{u}, \eta$.
- Much of the difficulty in taking the artificial pressure limit is controlling the oscillation defect measure for the fluid density $\varrho$.

# Oscillation Defect Measure and Strong Convergence

## Definition

Let $Q \subset \Omega$ and $q \geq 1$. Then

$$\mathbf{osc}_q[\varrho_\delta - \varrho](Q) := \sup_{k \geq 1} \left( \limsup_{\delta \to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}x \right).$$

Here, $\{T_k\}$ is a family of appropriately concave cutoff functions. Using these cutoff functions, we can control the oscillation defect measure and obtain strong convergence of the fluid density.

# Approximate Relative Entropy Inequality

- We formulate an approximate relative entropy inequality for each fixed $n, \varepsilon, \delta$.
- We define smooth functions $\mathbf{U}_m \in C^1([0, T]; X_m)$ zero on the boundary and positive $r_m, s_m$ on $[0, T] \times \overline{\Omega}$.
- We take $\mathbf{u}_n - \mathbf{U}_m$ as a test function on the Faedo-Galerkin approximate momentum equation and perform some calculations to obtain an approximate relative entropy inequality.
- We take the limits to obtain the relative entropy inequality.

# Relative Entropy Inequality

Regularity of $r, \mathbf{U}, s$ are imposed to ensure that all integrals in the formula for the relative entropy are defined.

$$
\begin{aligned}
& r \in C_{\text{weak}}([0,T]; L^{\gamma}(\Omega)) \\
& \mathbf{U} \in C_{\text{weak}}([0,T]; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \\
& \nabla_x \mathbf{U} \in L^2(0,T; L^2(\Omega;\mathbb{R}^{3\times 3})),\ \mathbf{U}|_{\partial\Omega} = 0 \\
& s \in C_{\text{weak}}([0,T]; L^1(\Omega)) \cap L^1(0,T; L^{6\gamma/\gamma-3}(\Omega)) \\
& \partial_t \mathbf{U} \in L^1(0,T; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3)) \\
& \nabla_x^2 \mathbf{U} \in L^1(0,T; L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^{3\times3\times3})) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^{3\times3\times3})) \\
& \partial_t P_F(r) \in L^1(0,T; L^{\gamma/\gamma-1}(\Omega)) \\
& \nabla_x P_F(r) \in L^1(0,T; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3)) \\
& \partial_t P_P(s) \in L^1(0,T; L^{\infty}(\Omega)) \cap L^{\infty}(0,T; L^{3/2}(\Omega)) \\
& \nabla_x P_P(s) \in L^{\infty}(0,T; L^3(\Omega;\mathbb{R}^3)) \\
& \nabla_x s \in L^{\infty}(0,T; L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma+3}(\Omega;\mathbb{R}^3)). \qquad (32)
\end{aligned}
$$

# Uniqueness of Weakly Dissipative Solutions

## Theorem (Weak-Strong Uniqueness)

*Assume* $\{\varrho, \mathbf{u}, \eta\}$ *is a weakly dissipative solution of the NSS system. Assume that* $\{r, \mathbf{U}, s\}$ *is a smooth solution of the NSS system with appropriate regularity with the same initial data. Then* $\{\varrho, \mathbf{u}, \eta\}$ *and* $\{r, \mathbf{U}, s\}$ *are identical.*

Note that the following hypotheses are imposed on the smooth solutions

$$\begin{aligned}
&\nabla_x r \in L^2(0, T; L^q(\Omega; \mathbb{R}^3)) \\
&\nabla_x^2 \mathbf{U} \in L^2(0, T; L^q(\Omega; \mathbb{R}^{3\times 3\times 3})) \\
&\alpha := \nabla_x s + s\nabla_x \Phi \in L^2(0, T; L^q(\Omega; \mathbb{R}^3))
\end{aligned} \tag{33}$$

where

$$q > \max\left\{3, \frac{3}{\gamma - 1}\right\}$$

The proof involves analysis bounding the remainder terms in terms of the relative entropy and using Gronwall's inequality on the result.

# Remarks

- The result can be generalized to unbounded spatial domains by creating a sequence of bounded domains and passing the limits through using the confinement hypotheses.
- This result does not show the existence of appropriately smooth $\{r, \mathbf{U}, s\}$, which is the focus of other work.

# Low Stratification

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{34}$$

$$\varepsilon^2[\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] + \nabla_x \left( a\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon \right)$$

$$= \varepsilon^2(\mu\Delta_x \mathbf{u}_\varepsilon + \lambda\nabla_x \,\mathrm{div}_x\, \mathbf{u}_\varepsilon) - \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x\Phi \tag{35}$$

$$\partial_t \eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon) - \varepsilon\, \mathrm{div}_x(\zeta\eta_\varepsilon\nabla_x\Phi) - D\Delta_x\eta_\varepsilon = 0 \tag{36}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta}\ln\eta_\varepsilon + \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi \,\mathrm{d}x$$

$$+ \int_\Omega D^2\frac{|\nabla_x\eta_\varepsilon|^2}{\zeta\eta_\varepsilon} + 2\varepsilon D\nabla_x\eta_\varepsilon\cdot\nabla_x\Phi + \varepsilon^2\zeta\eta_\varepsilon|\nabla_x\Phi|^2 \,\mathrm{d}x$$

$$+ \int_\Omega \varepsilon^2\mathbb{S}(\nabla_x\mathbf{u}_\varepsilon) : \nabla_x\mathbf{u}_\varepsilon \,\mathrm{d}x \le 0 \tag{37}$$

# Formal Evaluation of the Low Stratification Low Mach Number Limit

- Assume the following expansions:

$$\varrho_\varepsilon = \overline{\varrho} + \sum_{i=1}^{\infty} \varepsilon^i \varrho_\varepsilon^{(i)}$$

$$\eta_\varepsilon = \overline{\eta} + \sum_{i=1}^{\infty} \varepsilon^i \eta_\varepsilon^{(i)}$$

$$\mathbf{u}_\varepsilon = \overline{\mathbf{u}} + \sum_{i=1}^{\infty} \varepsilon^i \mathbf{u}_\varepsilon^{(i)}$$

- By considering the energy inequality, $\nabla_x \overline{\eta} = 0$, so $\overline{\eta} = \frac{1}{|\Omega|} \int_\Omega \eta_0(x) \, \mathrm{d}x$.
- By equating terms of order 1 in the momentum equation, $\nabla_x \left( a\overline{\varrho}^\gamma + \frac{D}{\zeta} \overline{\eta} \right) = 0$, implying $\overline{\varrho} = \frac{1}{|\Omega|} \int_\Omega \varrho_0(x) \, \mathrm{d}x$.
- Thus, $\overline{\mathbf{u}}$ satisfies the incompressibility condition $\mathrm{div}_x \, \overline{\mathbf{u}} = 0$.

# Low Stratification Limit

$$\overline{\eta} = \frac{1}{|\Omega|} \int_{\Omega} \eta_0(x) dx \tag{38}$$

$$\overline{\varrho} = \frac{1}{|\Omega|} \int_{\Omega} \varrho_0(x) dx \tag{39}$$

$$\mathrm{div}_x \, \overline{\mathbf{u}} = 0 \tag{40}$$

$$\overline{\varrho}[\partial_t \overline{\mathbf{u}} + \mathrm{div}_x(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}})] + \nabla_x \Pi = \mu \Delta_x \overline{\mathbf{u}} - (\beta r + \theta) \nabla_x \Phi \tag{41}$$

where $r, \theta$ satisfy

$$\nabla_x \left( a r^{\gamma} + \frac{D}{\zeta} \theta \right) = -(\beta \overline{\varrho} + \overline{\eta}) \nabla_x \Phi$$

# Low Stratification System Weak Formulation I

$\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ form a weak solution to the scaled low stratification equations if:

$\varrho_\varepsilon \geq 0$ and $\mathbf{u}_\varepsilon$ form a renormalized solution of the scaled continuity equation, i.e.,

$$\int_0^T \int_\Omega B(\varrho_\varepsilon)\partial_t\varphi + B(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi - b(\varrho_\varepsilon)\,\mathrm{div}_x\,\mathbf{u}_\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_\Omega B(\varrho_0)\varphi(0,\cdot)\,\mathrm{d}x \tag{42}$$

where $b \in L^\infty \cap C[0,\infty)$, $B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z$.
The scaled momentum balance holds in the sense of distributions:

$$\int_0^T \int_\Omega \varepsilon^2 \left(\varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \partial_t\mathbf{v} + \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x\mathbf{v}\right) + \left(p_F(\varrho_\varepsilon) + \frac{D}{\zeta}\eta_\varepsilon\right)\mathrm{div}_x\,\mathbf{v}\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \varepsilon^2 \left(\mu\nabla_x\mathbf{u}_\varepsilon\nabla_x\mathbf{v} + \lambda\,\mathrm{div}_x\,\mathbf{u}_\varepsilon\,\mathrm{div}_x\,\mathbf{v}\right) - \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x\Phi \cdot \mathbf{v}\,\mathrm{d}x\,\mathrm{d}t$$
$$- \varepsilon^2 \int_\Omega \mathbf{m}_0 \cdot \mathbf{v}(0,\cdot)\,\mathrm{d}x \tag{43}$$

# Low Stratification System Weak Formulation II

- $\eta_\varepsilon \geq 0$ is a weak solution of the scaled Smoluchowski equation:

$$\int_0^T \int_\Omega \eta_\varepsilon \partial_t \varphi + \eta_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi - \zeta \eta_\varepsilon \nabla_x \Phi \cdot \nabla_x \varphi - D \nabla_x \eta_\varepsilon \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$
$$= - \int_\Omega \eta_0 \varphi(0, \cdot) \, \mathrm{d}x \qquad (44)$$

- The energy inequality is satisfied:

$$\int_\Omega \frac{\varepsilon^2}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma - 1} \varrho_\varepsilon^\gamma + \frac{D}{\zeta} \eta_\varepsilon \ln \eta_\varepsilon + \varepsilon(\beta \varrho_\varepsilon + \eta_\varepsilon) \Phi \, \mathrm{d}x(T)$$
$$+ \int_0^T \int_\Omega \varepsilon^2 (\mu |\nabla_x \mathbf{u}_\varepsilon|^2 + \lambda |\operatorname{div}_x \mathbf{u}_\varepsilon|^2) \, \mathrm{d}x \, \mathrm{d}t$$
$$+ \int_0^T \int_\Omega \left| 2 \frac{D}{\sqrt{\zeta}} \nabla_x \sqrt{\eta_\varepsilon} + \varepsilon \sqrt{\zeta \eta_\varepsilon} \nabla_x \Phi \right|^2 \, \mathrm{d}x \, \mathrm{d}t$$
$$\leq \int_\Omega \frac{\varepsilon^2}{2} \varrho_0 |\mathbf{u}_0|^2 + \frac{a}{\gamma - 1} \varrho_0^\gamma + \frac{D}{\zeta} \eta_0 \ln \eta_0 + \varepsilon(\beta \varrho_0 + \eta_0) \Phi \, \mathrm{d}x \qquad (45)$$

# Target System

## Definition (Low Stratification Target System)

We say that $\{\overline{\mathbf{u}}, r, s\}$ solve the *low stratification target system* if

$$\mathrm{div}_x\, \overline{\mathbf{u}} = 0 \text{ weakly on } (0,T)\times\Omega,$$

$$\int_0^T \int_\Omega \overline{\varrho}\overline{\mathbf{u}} \cdot \partial_t \mathbf{v} + \overline{\varrho}\overline{\mathbf{u}} \otimes \overline{\mathbf{u}} : \nabla_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega (\mu \nabla_x \overline{\mathbf{u}} - (\beta r + s)\nabla_x \Phi) \cdot \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \overline{\varrho}\overline{\mathbf{u}} \cdot \mathbf{v}(0,\cdot) \, \mathrm{d}x,$$

for any divergence-free test function $\mathbf{v}$ and

$$r = -\frac{1}{a\gamma\overline{\varrho}^{\gamma-1}} \left[ (\beta\overline{\varrho} + \overline{\eta})\Phi + \frac{D}{\zeta} s \right]$$

weakly.

# Main Result I

## Theorem (Low Stratification Limit)

*Let* $(\Omega, \Phi)$ *satisfy the confinement hypothesis and for each* $\varepsilon > 0$, $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ *solves* (42)-(45). *Assume the initial data can be expressed as*

$$\varrho_\varepsilon(0,\cdot) = \varrho_{\varepsilon,0} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{\varepsilon,0},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{\varepsilon,0},\ \text{and } \eta_\varepsilon(0,\cdot) = \eta_{\varepsilon,0} = \overline{\eta} + \varepsilon \eta^{(1)}_{\varepsilon,0}.$$

*where* $\overline{\varrho}, \overline{\eta}$ *are the spatially uniform densities on* $\Omega$. *Assume also that as* $\varepsilon \to 0$,

$$\varrho^{(1)}_{\varepsilon,0} \rightharpoonup \varrho^{(1)}_0, \mathbf{u}_{\varepsilon,0} \rightharpoonup \overline{\mathbf{u}}_0, \eta^{(1)}_{\varepsilon,0} \rightharpoonup \eta^{(1)}_0$$

*weakly-*$*$ *in* $L^\infty(\Omega)$ *or* $L^\infty(\Omega;\mathbb{R}^3)$.

*Then up to a subsequence and letting* $q := \min\{\gamma, 2\}$,

$$\begin{aligned}
\varrho_\varepsilon &\to \overline{\varrho} \text{ in } C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^q(\Omega)) \\
\eta_\varepsilon &\to \overline{\eta} \text{ in } L^2(0,T;L^2(\Omega)) \\
\mathbf{u}_\varepsilon &\to \overline{\mathbf{u}} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))
\end{aligned}$$

# Main Result II

*and*

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \textit{weakly-}* \ \textit{ in } L^\infty(0, T; L^q(\Omega))$$

$$\eta_\varepsilon^{(1)} = \frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon} \to \eta^{(1)} \textit{ weakly in } L^2(0, T; L^2(\Omega))$$

*where* $\{\overline{\mathbf{u}}, \varrho^{(1)}, \eta^{(1)}\}$ *solve the target system mentioned previously.*

# Free Energy Inequality

Recasting the energy inequality using the free energy

$$E_F(\varrho) + E_P(\eta) := \frac{a}{\gamma - 1}\varrho^\gamma - (\varrho - \overline{\varrho})\frac{a\gamma}{\gamma - 1}\overline{\varrho}^{\gamma-1} - \frac{a}{\gamma - 1}\overline{\varrho}^\gamma + \frac{D}{\zeta}\eta \ln \eta - \frac{D}{\zeta}(\eta - \overline{\eta})(\ln \overline{\eta} + 1) - \frac{D}{\zeta}\overline{\eta} \ln \overline{\eta},$$

we obtain

$$\int_\Omega \frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(E_F(\varrho_\varepsilon) + E_P(\eta_\varepsilon)) + \frac{1}{\varepsilon}(\beta \varrho_\varepsilon + \eta_\varepsilon)\Phi \, \mathrm{d}x(T)$$
$$+ \int_0^T \int_\Omega \mu |\nabla_x \mathbf{u}_\varepsilon|^2 + \lambda |\operatorname{div}_x \mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left| \frac{2D\nabla_x \sqrt{\eta_\varepsilon}}{\sqrt{\zeta}} + \varepsilon \sqrt{\zeta \eta_\varepsilon}\nabla_x \Phi \right|^2 \, \mathrm{d}x \, \mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}(E_F(\varrho_0) + E_P(\eta_0)) + \frac{1}{\varepsilon}(\beta \varrho_0 + \eta_0)\Phi \, \mathrm{d}x \tag{46}$$

# Momentum Equation

By using the uniform bounds and Sobolev embeddings, $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ converges to a limit $\overline{\varrho \mathbf{u} \otimes \mathbf{u}}$. Thus, the momentum equation converges to becomes

$$\int_0^T \int_\Omega \overline{\varrho} \overline{\mathbf{u}} \cdot \partial_t \mathbf{v} + \overline{\varrho \mathbf{u} \otimes \mathbf{u}} : \nabla_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \mu \nabla_x \overline{\mathbf{u}} : \nabla_x \mathbf{v} - (\beta \varrho^{(1)} + \eta^{(1)}) \nabla_x \Phi \cdot \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \overline{\varrho_0} \overline{\mathbf{u}_0} \cdot \mathbf{v} \, \mathrm{d}x$$

By dividing (43) by $\varepsilon$ and taking $\varepsilon \to 0^+$, we have weakly

$$\varrho^{(1)} = -\frac{1}{a\gamma \overline{\varrho}^{\gamma-1}} \left[ (\beta \overline{\varrho} + \overline{\eta}) \Phi + \frac{D}{\zeta} \eta^{(1)} \right]$$

# Helmholtz Decomposition

Consider a vector $\mathbf{v} \in \mathbb{R}^3$. We can decompose the vector into a gradient part

$$\mathbf{H}^{\perp}[\mathbf{v}] := \nabla_x \Delta_x^{-1} \operatorname{div}_x \mathbf{v}$$

and a divergence-free part

$$\mathbf{H}[\mathbf{v}] := \mathbf{v} - \mathbf{H}^{\perp}[\mathbf{v}]$$

Note that the Helmholtz projections are continuous and linear.

# Convective Term I

We decompose the tensor $\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon$ using the Helmholtz projections into

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon + \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] + \mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^\perp[\mathbf{u}_\varepsilon]$$

Using the properities of the Helmholtz projections and the convergence results earlier,

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \mathbf{H}[\overline{\varrho}\overline{\mathbf{u}}] = \overline{\varrho}\overline{\mathbf{u}}$$

in $C_{\text{weak}}([0, T]; L^{2q/q+1}(\Omega; \mathbb{R}^3))$,
and $\mathbf{H}[\mathbf{u}_\varepsilon] \to \overline{\mathbf{u}}$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$, so

$$\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{u}_\varepsilon \to \overline{\varrho}\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}$$

$$\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}[\mathbf{u}_\varepsilon] \to 0$$

weakly in $L^2(0, T; L^{6q/4q+3}(\Omega; \mathbb{R}^{3\times 3}))$.

# Convective Term II

After some manipulations, the scaled NSS system becomes

$$\int_0^T \int_\Omega \varepsilon r_\varepsilon \partial_t \phi + \mathbf{V}_\varepsilon \cdot \nabla_x \phi \, dx \, dt = \int_0^T \int_\Omega \mathbf{h}_\varepsilon^2 \cdot \nabla_x \phi \, dx \, dt \tag{47}$$

$$\int_0^T \int_\Omega \varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \mathbf{v} + \omega r_\varepsilon \operatorname{div}_x \mathbf{v} \, dx \, dt$$
$$= \int_0^T \int_\Omega [\beta(\overline{\varrho} - \varrho_\varepsilon) + (\overline{\eta} - \eta_\varepsilon)] \nabla_x \Phi \cdot \mathbf{v} + h_\varepsilon^1 : \nabla_x \mathbf{v} - h_\varepsilon^3 \operatorname{div}_x \mathbf{v} \, dx \, dt \tag{48}$$

where

$$\mathbf{V}_\varepsilon := \varrho_\varepsilon \mathbf{u}_\varepsilon$$
$$r_\varepsilon := \varrho_\varepsilon^{(1)} + \frac{D}{a\gamma \overline{\varrho}^{\gamma-1} \zeta} \eta_\varepsilon^{(1)} + \frac{(\beta \overline{\varrho} + \overline{\eta})\Phi}{a\gamma \overline{\varrho}^{\gamma-1}}$$
$$\omega := a\gamma \overline{\varrho}^{\gamma-1}$$

and $h_\varepsilon^1$, $\mathbf{h}_\varepsilon^2$, and $h_\varepsilon^3$ are quantities converging to zero.

# Convective Term III

In light of (47)-(48), we consider the eigenvalue problem

$$-\Delta_x q = \Lambda q$$
$$\nabla_x q \cdot \mathbf{n}|_{\partial\Omega} = 0$$
$$-\Lambda = \frac{\lambda^2}{\omega}$$

with a countable system of eigenvalues $0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \ldots$ and eigenvectors $\{q_n\}_{n=0}^{\infty}$.

# Decomposition of $L^2(\Omega;\mathbb{R}^3)$

Defining

$$\mathbf{w}_{\pm n} := \pm i\sqrt{\frac{\omega}{\Lambda_n}}\nabla_x q_n$$

where $q_n$, $\Lambda_n$ are defined from the previous eigenvalue problem. Thus, we decompose the space

$$L^2(\Omega;\mathbb{R}^3) = L^2_\sigma(\Omega;\mathbb{R}^3) \oplus L^2_g(\Omega;\mathbb{R}^3)$$

where

$$L^2_g(\Omega;\mathbb{R}^3) := \text{closure}_{L^2}\left\{\text{span}\left\{\frac{-i}{\omega}\mathbf{w}_n\right\}_{n=1}^{\infty}\right\}$$
$$L^2_\sigma(\Omega;\mathbb{R}^3) := \text{closure}_{L^2}\{\mathbf{v} \in C_c^\infty(\Omega;\mathbb{R}^3) |\, \text{div}_x\, \mathbf{v} = 0\}$$

and define the projection

$$\mathbf{P}_M : L^2(\Omega;\mathbb{R}^3) \to \mathrm{span}\left\{\frac{-i}{\sqrt{\omega}}\mathbf{w}_n\right\}_{n\leq M}$$

Note that we define $\mathbf{H}_M^\perp := \mathbf{P}_M\mathbf{H}^\perp = \mathbf{H}^\perp\mathbf{P}_M$ since the operators $\mathbf{H}^\perp$ and $\mathbf{P}_M$ commute.

# Return to the Singular Term

Rewriting the singular term and noting convergences, the problem of showing the singular term converges weakly to a gradient reduces to showing

$$\int_0^T \int_\Omega \mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] : \nabla_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \to 0$$

for each fixed $M \in \mathbb{N}$ as $\varepsilon \to 0$.

# Concluding Remarks

- The mechanical relative entropy for the NSS system can be used to obtain a weak-strong uniqueness result by finding the relative entropy between a weakly-dissipative solution and a smooth solution.
- A modification of the mechanical relative entropy between a weak solution and a solution to a given target system is used to find uniform bounds to show the convergence of the weak solutions to the target system as the Mach number becomes small.
- Current work is investigating the use of the relative entropy to show the existence of measure-valued solutions to a corresponding model for inviscid fluids.

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