Weak-Strong Uniqueness of the Navier-Stokes-Smoluchowski System

Joshua Ballew

University of Maryland College Park Applied PDE RIT

March 4, 2013
Outline

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Fluid-particle interaction models are of interest to engineers and scientists studying biotechnology, medicine, waste-water recycling, mineral processing, and combustion theory.

The macroscopic model considered in this talk, the Navier-Stokes-Smoluchowski system, is formally derived from a Fokker-Planck type kinetic equation coupled with fluid equations. This coupling is from the mutual frictional forces between the particles and the fluid, assumed to follow Stokes’ Law.

The fluid is a viscous, Newtonian, compressible fluid.
Navier-Stokes-Smoluchowski System

\[ \partial_t \rho + \text{div}_x (\rho u) = 0 \]  \hspace{1cm} (1)

\[ \partial_t (\rho u) + \text{div}_x (\rho u \otimes u) + \nabla_x \left( a \rho^\gamma + \frac{D}{\zeta} \eta \right) = \text{div}_x S(\nabla_x u) - (\beta \rho + \eta) \nabla_x \Phi \]  \hspace{1cm} (2)

\[ \partial_t \eta + \text{div}_x (\eta u - \zeta \eta \nabla_x \Phi) = D \Delta_x \eta \]  \hspace{1cm} (3)

\[ \int_{\Omega} \frac{1}{2} \rho |u|^2 + \frac{a}{\gamma - 1} \rho^\gamma + \frac{D}{\zeta} \eta \ln \eta + (\beta \rho + \eta) \Phi \, dx(\tau) \]

\[ + \int_0^\tau \int_{\Omega} \mu |\nabla_x u|^2 + \lambda |\text{div}_x u|^2 + \left| \frac{2D}{\sqrt{\zeta}} \nabla_x \sqrt{\eta} + \sqrt{\zeta \eta} \nabla_x \Phi \right|^2 \, dx \, dt \]

\[ \leq \int_{\Omega} \frac{1}{2} \rho_0 |u_0|^2 + \frac{a}{\gamma - 1} \rho_0^\gamma + \frac{D}{\zeta} \eta_0 \ln \eta_0 + (\beta \rho_0 + \eta_0) \nabla_x \Phi \, dx \]  \hspace{1cm} (4)
Constitutive Relations and Boundary and Initial Conditions

Newtonian Condition for a Viscous Fluid:

\[
\mathbb{S}(\nabla_x u) := \mu(\nabla_x u + \nabla_x^T u) + \lambda \text{div}_x u I
\]

\[
\mu > 0, \quad \lambda + \frac{2}{3} \mu \geq 0
\]

Pressure Conditions:

\[
\gamma > \frac{3}{2}, \quad a > 0
\]

Boundary and Initial Conditions:

\[
u|_{\partial \Omega} = (D \nabla_x \eta + \zeta \eta \nabla_x \Phi) \cdot n|_{\partial \Omega} = 0
\]

\[
 \varrho_0 \in L^\gamma(\Omega) \cap L^1_+(\Omega)
\]

\[
 m_0 \in L^{\frac{6}{5}}(\Omega; \mathbb{R}^3) \cap L^1(\Omega; \mathbb{R}^3)
\]

\[
 \eta_0 \in L^2(\Omega) \cap L^1_+(\Omega)
\]

For the purposes of this talk, we take \(D, \zeta = 1\).
The cloud of particles is described by its distribution function $f_{\varepsilon}(t, x, \xi)$ on phase space, which is the solution to the dimensionless Vlasov-Fokker-Planck equation

$$\frac{\partial_t f_{\varepsilon}}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon} (\xi \cdot \nabla_x f_{\varepsilon} - \nabla_x \Phi \cdot \nabla_\xi f_{\varepsilon}) = \frac{1}{\varepsilon} \text{div}_\xi \left( (\xi - \sqrt{\varepsilon} u_{\varepsilon}) f + \nabla_\xi f_{\varepsilon} \right).$$

The friction force is assumed to follow Stokes law and thus is proportional to the relative velocity vector, i.e., is proportional to the fluctuations of the microscopic velocity $\xi \in \mathbb{R}^3$ around the fluid velocity field $u$. The RHS of the momentum equation in the Navier-Stokes system takes into account the action of the cloud of particles on the fluid through the forcing term

$$F_{\varepsilon} = \int_{\mathbb{R}^3} \left( \frac{\xi}{\sqrt{\varepsilon}} - u_{\varepsilon}(t, x) \right) f(t, x, \xi) \, d\xi.$$
The density of the particles $\eta_\varepsilon(t, x)$ is related to the probability distribution function $f_\varepsilon(t, x, \xi)$ through the relation

$$\eta_\varepsilon(t, x) = \int_{\mathbb{R}^3} f_\varepsilon(t, x, \xi) \, d\xi.$$
Confinement Hypotheses

Take $\Phi : \Omega \mapsto \mathbb{R}^+$ where $\Omega$ is a $C^{2,\nu}$ domain.

**Bounded Domain**

- $\Phi$ is bounded and Lipschitz on $\overline{\Omega}$.
- $\beta \neq 0$.
- The sub-level sets $[\Phi < k]$ are connected in $\Omega$ for all $k > 0$.

**Unbounded Domain**

- $\Phi \in W^{1,\infty}_{loc}(\Omega)$.
- $\beta > 0$.
- The sub-level sets $[\Phi < k]$ are connected in $\Omega$ for all $k > 0$.
- $e^{-\Phi/2} \in L^1(\Omega)$.
- $|\Delta_x \Phi(x)| \leq c_1|\nabla_x \Phi(x)| \leq c_2 \Phi(x)$ for $x$ with sufficiently large magnitude.
Carrillo et al. (2010) established the existence of renormalized weak solutions in the following sense:

**Definition**
Assume that $\Phi, \Omega$ satisfy the confinement hypotheses. Then $\{\varrho, u, \eta\}$ represent a *renormalized weak solution* to (1)-(4) if and only if

- $\varrho \geq 0$, $u$ represent a renormalized solution of (1),
- equations (2) and (3) are satisfied in the sense of distributions,
- inequality (4) is satisfied for all $\tau \in [0, T]$, and
- all the weak formulations are well-defined, that is, $\varrho \in C([0, T]; L^1(\Omega)) \cap L^\infty(0, T; L^\gamma(\Omega))$, $u \in L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^3))$, and $\eta \in L^2(0, T; L^3(\Omega)) \cap L^1(0, T; W^{1,\frac{3}{2}}(\Omega))$. 
Weak Existence

- This existence result of Carrillo et al. was established by implementing a time-discretization approximation supplemented with an artificial pressure approximation.
- Their paper also handles the case of unbounded domains and proves the convergence to a steady-state solution as $t \to \infty$. 
Entropy/Entropy Flux Pairs

For simplicity, consider the hyperbolic equation for a one-dimensional spatial domain equation

$$\partial_t U + \partial_x G(U) = 0$$  \hspace{1cm} (9)

Examples include the inviscid Burgers’ equation \((G(U) = \frac{1}{2} U^2)\). Consider functions \(E(U, x, t)\) and \(Q(U, x, t)\) such that

\[DQ = DEDG\]

\(E\) is called an entropy and \(Q\) and entropy flux for (9). Together, they are called an entropy/entropy flux pair.

If (9) has such a pair,

$$\partial_t E + \partial_x Q \leq 0.$$

For smooth solutions, the above inequality becomes an equality.
Relative Entropy

Consider

$$\partial_t U + \partial_x G(U) = 0$$

dAINED WITH AN ENTROPY/ENTROPY-FLUX PAIR \((E, Q)\). We define the *relative entropy* \(H(U|\bar{U})\) as

$$H(U|\bar{U}) := E(U) - E(\bar{U}) - D_E(\bar{U})(U - \bar{U}) \quad (10)$$

Note that this definition will only consider quadratic terms, but not linear terms in the entropy. We choose this definition because if we have that \(c_1, c_2 > 0\) and \(D^2E\) positive definite such that

$$c_1 \mathbb{I} \leq D^2E \leq c_2 \mathbb{I},$$

then there are \(c_3, c_4 > 0\) such that

$$c_3 |U - \bar{U}|^2 \leq H(U|\bar{U}) \leq c_4 |U - \bar{U}|^2.$$
Next, we define a stronger version of solution:

**Definition (Weakly Dissipative Solutions)**

\( \{ \rho, \mathbf{u}, \eta \} \) are called a *weak dissipative solution* to the NSS system if and only if

- \( \{ \rho, \mathbf{u}, \eta \} \) form a renormalized weak solution with the energy inequality

\[
\int_{\Omega} \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{a}{\gamma - 1} \rho^\gamma + \eta \ln \eta + \eta \Phi \, dx(\tau)
\]

\[
+ \int_0^\tau \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + |2\nabla_x \sqrt{\eta} + \sqrt{\eta} \nabla_x \Phi|^2 \, dx \, dt
\]

\[
\leq \int_{\Omega} \frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{a}{\gamma - 1} \rho_0^\gamma + \eta_0 \ln \eta_0 + \eta_0 \Phi \, dx
\]

\[
- \beta \int_0^\tau \int_{\Omega} \rho \mathbf{u} \cdot \nabla_x \Phi \, dx \, dt
\]

satisfied for all \( \tau \).
Weakly Dissipative Solutions II

Definition (Weakly Dissipative Solutions)

- for all suitably smooth solutions \( \{r, U, s\} \) of the NSS system, the following relative entropy inequality holds for all \( \tau \).

\[
\int_{\Omega} \frac{1}{2} \varrho |u - U|^2 + E_F(\varrho, r) + E_P(\eta, s) \, dx(\tau) \\
+ \int_0^\tau \int_{\Omega} [S(\nabla_x u) - S(\nabla_x U)] : \nabla_x (u - U) \, dx \, dt \\
\leq \int_{\Omega} \frac{1}{2} \varrho_0 |u_0 - U_0|^2 + E_F(\varrho_0, r_0) + E_P(\eta_0, s_0) \, dx \\
+ \int_0^\tau \mathcal{R}(\varrho, u, \eta, r, U, s) \, dt \tag{12}
\]
Remainder Term

The remainder term in (12) has the form

\[ R(\varrho, u, \eta, r, U, s) \]

\[ := \int_{\Omega} \text{div}_x (S(\nabla_x U)) \cdot (U - u) \, dx - \int_{\Omega} \varrho (\partial_t U + u \cdot \nabla_x U) \cdot (u - U) \, dx \]

\[ - \int_{\Omega} \partial_t P_F(r)(\varrho - r) + \nabla_x P_F(r) \cdot (\varrho u - rU) \, dx \]

\[ - \int_{\Omega} [\varrho(P_F(\varrho) - P_F(r)) - E_F(\varrho, r)] \text{div}_x U \, dx \]

\[ - \int_{\Omega} \partial_t P_P(s)(\eta - s) + \nabla_x P_P(s) \cdot (\eta u - sU) \, dx \]

\[ - \int_{\Omega} [\eta(P_P(\eta) - P_P(s)) - E_P(\eta, s)] \text{div}_x U \, dx \]

\[ - \int_{\Omega} \nabla_x (P_P(\eta) - P_P(s)) \cdot (\nabla_x \eta + \eta \nabla_x \Phi) \, dx \]

\[ - \int_{\Omega} \left[ (\beta \varrho + \eta) \nabla_x \Phi + \frac{\eta \nabla_x s}{s} \right] \cdot (u - U) \, dx \]
A three-level approximation scheme is employed

- Artificial pressure parameterized by small $\delta$
- Vanishing viscosity parameterized by small $\varepsilon$
- Faedo-Galerkin approximation where test functions for the momentum equation are taken from $n$-dimensional function spaces $X_n$ of smooth functions on $\overline{\Omega}$
Approximate System

\[
\partial_t \rho_n + \text{div}_x (\rho_n u_n) = \varepsilon \Delta_x \rho_n \quad (13)
\]

\[
\partial_t \eta_n + \text{div}_x (\eta_n u_n - \eta_n \nabla_x \Phi) = \Delta_x \eta_n \quad (14)
\]

\[
\int_{\Omega} \partial_t (\rho_n u_n) \cdot w \, dx = \int_{\Omega} \rho_n u_n \otimes u_n : \nabla_x w + (a \rho_n^\gamma + \eta_n + \delta \rho_n^\alpha) \text{div}_x w \, dx
\]

\[
- \int_{\Omega} S(\nabla_x u_n) : \nabla_x w + \varepsilon \nabla_x \rho_n \cdot \nabla_x u_n \cdot w \, dx - \int_{\Omega} (\beta \rho_n + \eta_n) \nabla_x \Phi \cdot w \, dx
\quad (15)
\]

with the additional conditions

\[
\nabla_x \rho_n \cdot n = 0 \text{ on } (0, T) \times \partial \Omega
\]

\[
u_n = (\nabla_x \eta_n + \eta_n \nabla_x \Phi) \cdot n = 0 \text{ on } (0, T) \times \partial \Omega
\]
Existence of Approximate Solutions

- Existence of $u_n$ is obtained from the Faedo-Galerkin approximation and an iteration argument in the spirit of Feireisl.
- $\rho_n, \eta_n$ obtained from $u_n$ using fixed point arguments in the spirit of Ladyzhenskaya.
Approximate Energy Inequality

Using \( u_n \) as a test function in (15) and some straight-forward manipulations:

\[
\int_{\Omega} \frac{1}{2} \varrho_n |u_n|^2 + \frac{a}{\gamma - 1} \varrho_n^\gamma + \frac{\delta}{\alpha - 1} \varrho_n^\alpha + \eta_n \ln \eta_n + \eta_n \Phi \, dx(\tau)
\]

\[
+ \int_0^\tau \int_{\Omega} \mathcal{S}(\nabla_x u_n) : \nabla_x u_n + |2\nabla_x \sqrt{\eta_n} + \sqrt{\eta_n} \nabla_x \Phi|^2 \, dx \, dt
\]

\[
+ \varepsilon \int_0^\tau \int_{\Omega} |\nabla_x \varrho_n|^2 (a \gamma \varrho_n^{\gamma - 2} + \delta a \varrho_n^{\alpha - 2}) \, dx \, dt
\]

\[
\leq \int_{\Omega} \frac{1}{2} \varrho_{0,\delta} |u_{0,\delta}|^2 + \frac{a}{\gamma - 1} \varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha - 1} \varrho_{0,\delta}^\alpha + \eta_{0,\delta} \ln \eta_{0,\delta} + \eta_{0,\delta} \Phi \, dx
\]

\[
- \beta \int_0^\tau \int_{\Omega} \varrho_n u_n \cdot \nabla_x \Phi \, dx \, dt
\]

(16)
Uniform Bounds

From the energy inequality, we find that

\[
\begin{align*}
\{u\}_{n,\varepsilon,\delta} & \in _b L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^3)) \\
\{\sqrt{\rho u}\}_{n,\varepsilon,\delta} & \in _b L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)) \\
\{\rho\}_{n,\varepsilon,\delta} & \in _b L^\infty(0, T; L^\gamma(\Omega)) \\
\{\eta \ln \eta\}_{n,\varepsilon,\delta} & \in _b L^\infty(0, T; L^1(\Omega)) \\
\{\nabla x \sqrt{\eta}\}_{n,\varepsilon,\delta} & \in _b L^2(0, T; L^2(\Omega; \mathbb{R}^3)) \\
\{\eta\}_{n,\varepsilon,\delta} & \in _b L^2(0, T; W^{1,\frac{3}{2}}(\Omega))
\end{align*}
\]
Faedo-Galerkin Limit I

- From the approximate energy balance, the term

$$
\varepsilon \delta \int_0^T \int_{\Omega} |\nabla_{x} \varrho_n|^2 \varrho_n^{\alpha-2} \ dx \ dt
$$

is bounded independently of $n$. Thus by Poincaré's inequality,

$$\{ \varrho \}_n \in_b L^2(0, T; W^{1,2}(\Omega)).$$

- From this, $\nabla_{x} \varrho_n \cdot u_n \in_b L^1(0, T; L^{3/2}(\Omega))$. To get higher time integrability, multiply (13) by $G'(\varrho_n)$ where $G(\varrho_n) := \varrho_n \ln \varrho_n$. Then

$$
\varepsilon \int_0^T \int_{\Omega} \frac{|\nabla_{x} \varrho_n|^2}{\varrho_n} \ dx \ dt
$$

is bounded independently of $n$. Using Hölder’s and interpolation,

$$\{ \nabla_{x} \varrho_n \cdot u_n \}_n \in_b L^q(0, T; L^p(\Omega))$$

for some $p \in (1, \frac{3}{2})$ and $q \in (1, 2)$. Thus, $\varrho_\varepsilon, u_\varepsilon$ obey

$$\partial_t \varrho_\varepsilon + \text{div}_x(\varrho_\varepsilon u_\varepsilon) = \varepsilon \Delta_{x} \varrho_\varepsilon.$$
Faedo-Galerkin Limit II

- Strong convergence of $\nabla_x \varrho_n \to \nabla_x \varrho_\varepsilon$ follows from letting $G(z) = z^2$.
- Similar techniques show convergence of $\eta_n \to \eta_\varepsilon$ and $\nabla_x \eta_n \to \nabla_x \eta_\varepsilon$ to allow
  \[ \partial_t \eta_\varepsilon + \text{div}_x (\eta_\varepsilon u_\varepsilon - \eta_\varepsilon \nabla_x \Phi) = \Delta_x \eta_\varepsilon. \]
- Terms in the momentum equation converge as we want using the bounds and the above convergences, except for the convective term.
- Convergence of the convective term $\varrho_n u_n \otimes u_n$ in $L^q((0, T) \times \Omega; \mathbb{R}^3)$ follows from convergence of $\varrho_n u_n$ and Arzela-Ascoli.

The following lemma is of use throughout the analysis for convergence of the $\eta$ terms:

**Lemma (Simon)**

Let $X \subset B \subset Y$ be Banach spaces with $X \subset B$ compactly. Then, for $1 \leq p < \infty$, $\{v : v \in L^p(0, T; X), v_t \in L^1(0, T; Y)\}$ is compactly embedded in $L^p(0, T; B)$.

Thus, $\{\eta\}_{n,\varepsilon} \to \eta_\delta$ in $L^2(0, T; L^3(\Omega))$. 
Vanishing Viscosity Approximation I

\[ \partial_t \rho_\varepsilon + \text{div}_x (\rho_\varepsilon u_\varepsilon) = \varepsilon \Delta_x \rho_\varepsilon \tag{17} \]

\[ \partial_t \eta_\varepsilon + \text{div}_x (\eta_\varepsilon u_\varepsilon - \eta_\varepsilon \nabla_x \Phi) = \Delta_x \eta_\varepsilon \tag{18} \]

\[ \int_\Omega \partial_t (\rho_\varepsilon u_\varepsilon) \cdot w \, dx = \int_\Omega \rho_\varepsilon u_\varepsilon \otimes u_\varepsilon : \nabla_x w + (a_\rho^\gamma + \eta_\varepsilon + \delta_\rho^\alpha) \text{div}_x w \, dx \]

\[ - \int_\Omega \mathcal{S}(\nabla_x u_\varepsilon) : \nabla_x w + \varepsilon \nabla_x \rho_\varepsilon \cdot \nabla_x u_\varepsilon \cdot w \, dx - \int_\Omega (\beta_\rho + \eta_\varepsilon) \nabla_x \Phi \cdot w \, dx \tag{19} \]

\[ \nabla_x \rho_\varepsilon \cdot n = 0 \]

\[ u_\varepsilon|_{\partial\Omega} = (\nabla_x \eta_\varepsilon + \eta_\varepsilon \nabla_x \Phi) \cdot n|_{\partial\Omega} = 0 \]
Vanishing Viscosity Approximation II

\[ \int_{\Omega} \frac{1}{2} \rho_{\varepsilon} |u_{\varepsilon}|^2 + \frac{a}{\gamma - 1} \rho_{\varepsilon}^\gamma + \frac{\delta}{\alpha - 1} \rho_{\varepsilon}^\alpha + \eta_{\varepsilon} \ln \eta_{\varepsilon} + \eta_{\varepsilon} \Phi \ dx(\tau) \]
+ \int_{0}^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x u_{\varepsilon}) : \nabla_x u_{\varepsilon} + |2 \nabla_x \sqrt{\eta_{\varepsilon}} + \sqrt{\eta_{\varepsilon}} \nabla_x \Phi|^2 \ dx \ dt
+ \varepsilon \int_{0}^{\tau} \int_{\Omega} |\nabla_x \rho_{\varepsilon}|^2 (a \gamma \rho_{\varepsilon}^{\gamma - 2} + \delta a \rho_{\varepsilon}^{\alpha - 2}) \ dx \ dt
\leq \int_{\Omega} \frac{1}{2} \rho_{0,\delta} |u_{0,\delta}|^2 + \frac{a}{\gamma - 1} \rho_{0,\delta}^\gamma + \frac{\delta}{\alpha - 1} \rho_{0,\delta}^\alpha + \eta_{0,\delta} \ln \eta_{0,\delta} + \eta_{0,\delta} \Phi \ dx
- \beta \int_{0}^{\tau} \int_{\Omega} \rho_{\varepsilon} u_{\varepsilon} \cdot \nabla_x \Phi \ dx \ dt \] (20)
We begin by using the uniform bounds and obtaining weak limits $\rho_\delta, u_\delta, \eta_\delta$.

We show that since $\sqrt{\varepsilon} \nabla x \rho_\varepsilon \to 0$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$, and using Arzela-Ascoli, we have that $\rho_\delta, u_\delta$ solve the continuity equation weakly.

Similar analysis shows that $\eta_\delta, u_\delta$ solve the Smoluchowski equation weakly.
Convergence of the momentum equation is fairly straight-forward except for the pressure-related terms. Using the Bogovskii operator (analogous to an inverse divergence operator) and an appropriate test function, we find that $a \varrho^\gamma + \eta + \delta \varrho^\alpha$ has a weak limit.

To show this weak limit is $a \varrho^\gamma + \eta + \delta \varrho^\alpha$, we have to show the strong convergence of the fluid density (strong convergence of the particle density follows from the lemma of Simon).

This is obtained by using the test function $\psi(t)\zeta(x)\varphi_1(x)$ where $\psi \in C^\infty_c(0, T), \zeta \in C^\infty_c(\Omega), \varphi_1(x) := \nabla_x \Delta_x^{-1}(1_\Omega \varrho)$, and analysis involving the double Reisz transform and the Div-Curl Lemma.
Artificial Pressure Approximation I

\[ \int_0^T \int_{\Omega} \rho_\delta B(\rho_\delta) (\partial_t \phi + u_\delta \cdot \nabla_x \phi) \, dx \, dt + \int_{\Omega} \rho_{0,\delta} B(\rho_{0,\delta}) \phi(0, \cdot) \, dx \]

\[ = \int_0^T \int_{\Omega} b(\rho_\delta) \text{div}_x u_\delta \phi \, dx \, dt \tag{21} \]

\[ \int_0^T \int_{\Omega} \eta_\delta \partial_t \phi + (\eta_\delta u_\delta - \eta_\delta \nabla_x \Phi - \nabla_x \eta_\delta) \cdot \nabla_x \phi \, dx \, dt = -\int_{\Omega} \eta_{0,\delta} \phi(0, \cdot) \, dx \tag{22} \]

\[ \int_{\Omega} \partial_t (\rho_\delta u_\delta) w \, dx = \int_{\Omega} \rho_\delta u_\delta \otimes u_\delta : \nabla_x w + (a \rho^\gamma_\delta + \eta_\delta + \delta \rho^\alpha_\delta) \text{div}_x w \, dx \]

\[ - \int_{\Omega} \mathbb{S}(\nabla_x u_\delta) : \nabla_x w \, dx - \int_{\Omega} (\beta \rho_\delta + \eta_\delta) \nabla_x \Phi \cdot w \, dx \tag{23} \]
Artificial Pressure Approximation II

\[
\int_{\Omega} \frac{1}{2} \rho_{\delta} |\mathbf{u}_{\delta}|^2 + \frac{a}{\gamma - 1} \rho_{\delta}^\gamma + \frac{\delta}{\alpha - 1} \rho_{\delta}^\alpha + \eta_{\delta} \ln \eta_{\delta} + \eta_{\delta} \Phi \, d\mathbf{x}(\tau)
\]

\[
+ \int_0^\tau \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}_{\delta}) : \nabla_x \mathbf{u}_{\delta} + |2\nabla_x \sqrt{\eta_{\delta}} + \sqrt{\eta_{\delta}} \nabla_x \Phi|^2 \, d\mathbf{x} \, dt
\]

\[
\leq \int_{\Omega} \frac{1}{2} \rho_{0,\delta} |\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma - 1} \rho_{0,\delta}^\gamma + \frac{\delta}{\alpha - 1} \rho_{0,\delta}^\alpha + \eta_{0,\delta} \ln \eta_{0,\delta} + \eta_{0,\delta} \Phi \, d\mathbf{x}
\]

\[
- \beta \int_0^\tau \int_{\Omega} \rho_{\delta} \mathbf{u}_{\delta} \cdot \nabla_x \Phi \, d\mathbf{x} \, dt \tag{24}
\]
Artificial Pressure Limit

- Again, from uniform bounds, we are able to obtain the existence of weak limits $\rho, u, \eta$.
- Much of the difficulty in taking the artificial pressure limit is controlling the oscillation defect measure for the fluid density $\rho$. 

Definition
Let $Q \subset \Omega$ and $q \geq 1$. Then

$$\text{osc}_q[\varrho_\delta - \varrho](Q) := \sup_{k \geq 1} \left( \limsup_{\delta \to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, dx \right).$$

Here, $\{T_k\}$ is a family of appropriately concave cutoff functions. Using these cutoff functions, we can control the oscillation defect measure and obtain strong convergence of the fluid density.
Approximate Relative Entropy Inequality

- We formulate an approximate relative entropy inequality for each fixed \( n, \varepsilon, \delta \).
- We define smooth functions \( U_m \in C^1([0, T]; X_m) \) zero on the boundary and positive \( r_m, s_m \) on \([0, T] \times \Omega\).
- We take \( u_n - U_m \) as a test function on the Faedo-Galerkin approximate momentum equation and perform some calculations to obtain an approximate relative entropy inequality.
- We take the limits to obtain the relative entropy inequality.
Relative Entropy Inequality

Regularity of $r, U, s$ are imposed to ensure that all integrals in the formula for the relative entropy are defined.

\[
\begin{align*}
  r &\in C_{\text{weak}}([0, T]; L^\gamma(\Omega)) \\
  U &\in C_{\text{weak}}([0, T]; L^{2\gamma/\gamma-1}(\Omega; \mathbb{R}^3)) \\
  \nabla_x U &\in L^2(0, T; L^2(\Omega; \mathbb{R}^{3\times3})), \ U|_{\partial\Omega} = 0 \\
  s &\in C_{\text{weak}}([0, T]; L^1(\Omega)) \cap L^1(0, T; L^{6\gamma/\gamma-3}(\Omega)) \\
  \partial_t U &\in L^1(0, T; L^{2\gamma/\gamma-1}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; L^{6\gamma/5\gamma-6}(\Omega; \mathbb{R}^3)) \\
  \nabla_x U &\in L^1(0, T; L^{2\gamma/\gamma+1}(\Omega; \mathbb{R}^{3\times3\times3})) \cap L^2(0, T; L^{6\gamma/5\gamma-6}(\Omega; \mathbb{R}^{3\times3\times3})) \\
  \partial_t P_F(r) &\in L^1(0, T; L^{\gamma/\gamma-1}(\Omega)) \\
  \nabla_x P_F(r) &\in L^1(0, T; L^{2\gamma/\gamma-1}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; L^{6\gamma/5\gamma-6}(\Omega; \mathbb{R}^3)) \\
  \partial_t P_F(s) &\in L^1(0, T; L^{\infty}(\Omega)) \cap L^\infty(0, T; L^{3/2}(\Omega)) \\
  \nabla_x P_F(s) &\in L^{\infty}(0, T; L^{3}(\Omega; \mathbb{R}^3)) \\
  \nabla_x s &\in L^{\infty}(0, T; L^{2}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; L^{6\gamma/5\gamma+3}(\Omega; \mathbb{R}^3)).
\end{align*}
\]
Uniqueness of Weakly Dissipative Solutions

Theorem (Weak-Strong Uniqueness)

Assume \( \{ \varrho, u, \eta \} \) is a weakly dissipative solution of the NSS system. Assume that \( \{ r, U, s \} \) is a smooth solution of the NSS system with appropriate regularity with the same initial data. Then \( \{ \varrho, u, \eta \} \) and \( \{ r, U, s \} \) are identical.

Note that the following hypotheses are imposed on the smooth solutions

\[
\nabla_x r \in L^2(0, T; L^q(\Omega; \mathbb{R}^3)) \\
\nabla_x^2 U \in L^2(0, T; L^q(\Omega; \mathbb{R}^{3 \times 3 \times 3})) \\
\alpha := \nabla_x s + s \nabla_x \Phi \in L^2(0, T; L^q(\Omega; \mathbb{R}^3))
\]

where

\[
q > \max \left\{ 3, \frac{3}{\gamma - 1} \right\}
\]

The proof involves analysis bounding the remainder terms in terms of the relative entropy and using Gronwall’s inequality on the result.
The result can be generalized to unbounded spatial domains by creating a sequence of bounded domains and passing the limits through using the confinement hypotheses.

This result does not show the existence of appropriately smooth \( \{r, U, s\} \), which is the focus of current work.

First, existence of local strong solutions for appropriate initial data will be shown along the lines of Cho and Kim.

Second, appropriate blow-up conditions will be formulated that will enable us to extend the local result to a global result in the style of Fan, Jian, and Ou.
Local Existence of Strong Solutions

Proof for local-in-time existence requires the following regularity on the initial data with \( q \in (3, 6] \)

\[
\begin{align*}
\rho_0 &\in W^{1,q}(\Omega) \\
u_0 &\in W_0^{1,2}(\Omega; \mathbb{R}^3) \cap W^{2,2}(\Omega; \mathbb{R}^3) \\
\eta_0 &\in W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)
\end{align*}
\]

(27)

and a vector field \( h \in L^2(\Omega; \mathbb{R}^3) \) satisfying the compatibility conditions

\[
\begin{align*}
\Phi &\in W^{2,2}(\Omega) \\
\sqrt{\rho_0} h &= \nabla_x (a \rho_0^\gamma + \eta_0) - \text{div}_x S(\nabla_x u_0) + \eta_0 \nabla_x \Phi
\end{align*}
\]

(28)
Theorem (Local In Time Existence)

Consider the NSS system (1)-(3) with the boundary conditions (5), initial conditions (27) and compatibility conditions (28). Then there exists a unique solution \( \{\varrho, u, \eta\} \) such that

\[
\begin{align*}
\varrho & \in C([0, T]; W^{1,q}(\Omega)) \\
\varrho_t & \in C([0, T]; L^q(\Omega)) \\
\mathbf{u} & \in C([0, T]; W^{1,2}_0(\Omega; \mathbb{R}^3) \cap W^{2,2}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{2,q}(\Omega; \mathbb{R}^3)) \\
\mathbf{u}_t & \in L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^3)) \\
\eta & \in C([0, T]; W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0, T; W^{2,q}(\Omega)) \\
\eta_t & \in L^2(0, T; W^{1,2}_0(\Omega)).
\end{align*}
\]

for some finite \( T > 0 \).
Linear System

Analysis for local existence of strong solutions uses existence and estimates on solutions to the *linear NSS system*

\[
\begin{align*}
\partial_t \rho + \text{div}_x (\rho \mathbf{v}) &= 0 \quad (29) \\
\partial_t (\rho \mathbf{u}) + \text{div}_x (\rho \mathbf{v} \otimes \mathbf{u}) + \nabla_x (a \rho^\gamma + \eta) &= \mu \Delta_x \mathbf{u} + \lambda \nabla_x \text{div}_x \mathbf{u} - (\beta \rho + \eta) \nabla_x \Phi \quad (30) \\
\partial_t \eta + \text{div}_x (\eta \mathbf{v} - \eta \nabla_x \Phi) - \Delta_x \eta &= 0 \quad (31)
\end{align*}
\]

where

\[
\mathbf{v} \in C([0, T]; W_{0}^{1,2}(\Omega; \mathbb{R}^3) \cap W^{2,2}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{2,q}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{2,q}(\Omega; \mathbb{R}^3))
\]

\[
\mathbf{v}_t \in L^2(0, T; W_{0}^{1,2}(\Omega; \mathbb{R}^3)).
\]

We also assume

\[
0 < \delta \leq \rho_0
\]

to approximate the initial fluid density with one that does not have a vacuum.
Linear Approximation Existence

Using the method of characteristics and classical results on parabolic equations, we obtain the existence of solutions \( \{\varrho, u, \eta\} \) to (29)-(31) such that for some \( T > 0 \),

\[
\begin{align*}
\varrho & \in C([0, T]; W^{1,q}(\Omega)), \quad \varrho_t \in C([0, T]; L^q(\Omega)) \\
\eta & \in C([0, T]; W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0, T; W^{2,q}(\Omega)) \\
\eta_t & \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; W^{1,2}_0(\Omega)) \\
\eta_{tt} & \in L^2(0, T; W^{-1,2}(\Omega)) \\
u & \in C([0, T]; W^{1,2}_0(\Omega; \mathbb{R}^3) \cap W^{2,2}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{2,q}(\Omega; \mathbb{R}^3)) \\
u_t & \in C([0, T]; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{1,2}_0(\Omega; \mathbb{R}^3)) \\
u_{tt} & \in L^2(0, T; W^{-1,2}(\Omega; \mathbb{R}^3)).
\end{align*}
\]
Uniform Bounds

In order to be able to pass through the limit of $\delta \to 0$, we obtain the bounds uniform in $\delta$ on various quantities. Key among them are the ones on fluid density $\varrho$ and pressure $a\varrho^\gamma + \eta$ below.

\[
\| \varrho(t) \|_{W^{1,q}(\Omega)} \leq Cc_0 \\
\| \varrho_t(t) \|_{L^q(\Omega)} \leq Cc_2 \\
P(\varrho, \eta)(t) \text{ is continuous on } \Omega \\
\| \nabla_x P(\varrho, \eta)(t) \|_{L^q(\Omega; \mathbb{R}^3)} \leq Cc_0 + c_g \\
\| \partial_t P(\varrho, \eta)(t) \|_{L^2(\Omega)} \leq Cc_2 + c_g
\]

The pressure bounds are used to obtain $\delta$-independent bounds on $u$, and then the analysis follows that of Cho and Kim.
Because of the uniform bounds, we can take $\delta \to 0$, eliminating the positive lower bound for $\rho_0$.

- For each $\delta$, a positive initial density $\rho_0^\delta := \rho_0 + \delta$ and an approximation for $h$, $h^\delta$ is defined. Because of the uniform-in-$\delta$ bounds, we find a solution $\{\rho, u, \eta\}$ to the linear problem.

- These solutions are shown to converge to a solution of the linear problem with the following regularity.
Regularity of Linear-Vacuum System Solutions

\[ \rho \in C([0, T]; W^{1,q}(\Omega)), \quad \rho_t \in C([0, T]; L^q(\Omega)) \]
\[ \eta \in C([0, T]; W^{1,2}_0(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0, T; W^{2,q}(\Omega)) \]
\[ \eta_t \in L^2(0, T; W^{1,2}_0(\Omega)) \]
\[ \mathbf{u} \in C([0, T]; W^{1,2}_0(\Omega; \mathbb{R}^3) \cap W^{2,2}(\Omega; \mathbb{R}^3)) \cap L^2(0, T; W^{2,q}(\Omega; \mathbb{R}^3)) \]
\[ \eta_t \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)) \]
\[ \sqrt{\rho} \mathbf{u}_t \in L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)). \]  \((32)\)
Solutions to Nonlinear System

To find solutions to the nonlinear system, we find a sequence of solutions to the linear system with a sequence of functions \( \{v^k\} \) defined inductively.

- \( v^0 \) solves the initial value problem

\[
\partial_t w - \Delta_x w = 0, \quad w(0, \cdot) = u_0
\]

- \( v^{k+1} = u^k \) where \( u^k \) solves the linear system using \( v^k \) for \( v \).

- Using this induction, we get a sequence of solutions \( \{\varrho^k, u^k, \eta^k\} \) that converge to some \( \{\varrho, u, \eta\} \) which are smooth and solve the nonlinear system (1)-(3) for some finite time \( T > 0 \).

- Combined with the weak-strong uniqueness result, we know that if the initial data have compatibility conditions (27)-(28), then there is a unique smooth solution for some finite time.
Conclusion

- We have given another proof of the existence of renormalized solutions to the NSS system, albeit with a slightly different energy inequality. The key aspect of this class of solution is that it obeys the relative entropy inequality.
- If there is a solution with regularity given in (25)-(26), then there is only one weakly dissipative solution.
- If the initial data satisfy (27) and (28), then there is a unique strong solution for finite time.
- It remains to develop blow-up conditions for the NSS system.
References I


