Weak-Strong Uniqueness of the Navier-Stokes-Smoluchowski System

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Outline

Fluid-Particle Interaction

- Fluid-particle interaction models are of interest to engineers and scientists studying biotechnolgy, medicine, waste-water recycling, mineral processing, and combustion theory.
- The macroscopic model considered in this talk, the Navier-Stokes-Smoluchowski system, is formally derived from a Fokker-Planck type kinetic equation coupled with fluid equations.
- This coupling is from the mutual frictional forces between the particles and the fluid, assumed to follow Stokes' Law.
- The fluid is a viscous, Newtonian, compressible fluid.

Navier-Stokes-Smoluchowski System

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x \left(a\varrho^\gamma + \frac{D}{\zeta}\eta \right)$$
$$= \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) - (\beta\varrho + \eta)\nabla_x \Phi \tag{2}$$

$$\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u} - \zeta \eta \nabla_x \Phi) = D\Delta_x \eta \tag{3}$$

$$\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + \frac{D}{\zeta}\eta \ln \eta + (\beta\varrho + \eta)\Phi \ \mathrm{d}x(\tau)$$
$$+ \int_0^\tau \int_\Omega \mu|\nabla_x \mathbf{u}|^2 + \lambda|\mathrm{div}_x \mathbf{u}|^2 + \left| \frac{2D}{\sqrt{\zeta}}\nabla_x \sqrt{\eta} + \sqrt{\zeta\eta}\nabla_x \Phi \right|^2 \ \mathrm{d}x \ \mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma - 1}\varrho_0^\gamma + \frac{D}{\zeta}\eta_0 \ln \eta_0 + (\beta\varrho_0 + \eta_0)\nabla_x \Phi \ \mathrm{d}x \tag{4}$$

Constitutive Relations and Boundary and Initial Conditions

Newtonian Condition for a Viscous Fluid:

$$\mathbb{S}(\nabla_x \mathbf{u}) := \mu(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u}) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

$$\mu > 0,\ \lambda + \frac{2}{3}\mu \geq 0$$

Pressure Conditions:

$$\gamma > \frac{3}{2},\ a > 0$$

Boundary and Initial Conditions:

$$\mathbf{u}|_{\partial\Omega} = (D\nabla_x \eta + \zeta\eta\nabla_x \Phi) \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{5}$$

$$\varrho_0 \in L^\gamma(\Omega) \cap L^1_+(\Omega) \tag{6}$$

$$\mathbf{m}_0 \in L^{\frac{6}{5}}(\Omega; \mathbb{R}^3) \cap L^1(\Omega; \mathbb{R}^3) \tag{7}$$

$$\eta_0 \in L^2(\Omega) \cap L^1_+(\Omega) \tag{8}$$

For the purposes of this talk, we take D, $\zeta = 1$.

Smoluchowski Equation and Vlasov-Fokker-Planck Equation I

The cloud of particles is described by its distribution function $f_\varepsilon(t, x, \xi)$ on phase space, which is the solution to the dimensionless Vlasov-Fokker-Planck equation

$$\partial_t f_\varepsilon + \frac{1}{\sqrt{\varepsilon}} \left(\xi \cdot \nabla_x f_\varepsilon - \nabla_x \Phi \cdot \nabla_\xi f_\varepsilon \right) = \frac{1}{\varepsilon} \mathrm{div}_\xi \left(\left(\xi - \sqrt{\varepsilon} u_\varepsilon \right) f + \nabla_\xi f_\varepsilon \right).$$

The friction force is assumed to follow Stokes law and thus is proportional to the relative velocity vector, i.e., is proportional to the fluctuations of the microscopic velocity $\xi \in \mathbb{R}^3$ around the fluid velocity field $\mathbf{u}$. The RHS of the momentum equation in the Navier-Stokes system takes into account the action of the cloud of particles on the fluid through the forcing term

$$F_\varepsilon = \int_{\mathbb{R}^3} \left(\frac{\xi}{\sqrt{\varepsilon}} - u_\varepsilon(t, x) \right) f(t, x, \xi) \, \mathrm{d}\xi.$$

Smoluchowski Equation and Vlasov-Fokker-Planck Equation II

The density of the particles $\eta_\varepsilon(t,x)$ is related to the probability distribution function $f_\varepsilon(t,x,\xi)$ through the relation

$$\eta_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon(t,x,\xi)\, \mathrm{d}\xi.$$

Confinement Hypotheses

Take $\Phi : \Omega \mapsto \mathbb{R}^+$ where Ω is a $C^{2,\nu}$ domain.
Bounded Domain

- Φ is bounded and Lipschitz on $\overline{\Omega}$.
- $\beta \neq 0$.
- The sub-level sets $[\Phi < k]$ are connected in Ω for all $k > 0$.

Unbounded Domain

- $\Phi \in W^{1,\infty}_{loc}(\Omega)$.
- $\beta > 0$.
- The sub-level sets $[\Phi < k]$ are connected in Ω for all $k > 0$.
- $e^{-\Phi/2} \in L^1(\Omega)$.
- $|\Delta_x \Phi(x)| \leq c_1 |\nabla_x \Phi(x)| \leq c_2 \Phi(x)$ for x with sufficiently large magnitude.

Weak Formulation

Carrillo *et al.* (2010) established the existence of renormalized weak solutions in the following sense:

Definition

Assume that Φ, Ω satisfy the confinement hypotheses. Then $\{\varrho, \mathbf{u}, \eta\}$ represent a *renormalized weak solution* to (1)-(4) if and only if

- $\varrho \geq 0, \mathbf{u}$ represent a renormalized solution of (1),
- equations (2) and (3) are satisfied in the sense of distributions,
- inequality (4) is satisfied for all $\tau \in [0, T]$, and
- all the weak formulations are well-defined, that is, $\varrho \in C([0, T]; L^1(\Omega)) \cap L^\infty(0, T; L^\gamma(\Omega))$, $\mathbf{u} \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^3))$, and $\eta \in L^2(0, T; L^3(\Omega)) \cap L^1(0, T; W^{1,\frac{3}{2}}(\Omega))$.

Weak Existence

- This existence result of Carrillo et al. was established by implementing a time-discretization approximation supplemented with an artificial pressure approximation.
- Their paper also handles the case of unbounded domains and proves the convergence to a steady-state solution as $t \to \infty$.

Entropy/Entropy Flux Pairs

For simplicity, consider the hyperbolic equation for a one-dimensional spatial domain equation

$$\partial_t U + \partial_x G(U) = 0 \tag{9}$$

Examples include the inviscid Burgers' equation ($G(U) = \frac{1}{2}U^2$). Consider functions $\mathcal{E}(U, x, t)$ and $Q(U, x, t)$ such that

$$DQ = D\mathcal{E}DG$$

$\mathcal{E}$ is called an *entropy* and Q and *entropy flux* for (9). Together, they are called an *entropy/entropy flux pair*.
If (9) has such a pair,

$$\partial_t \mathcal{E} + \partial_x Q \leq 0.$$

For smooth solutions, the above inequality becomes an equality.

Relative Entropy

Consider

$$\partial_t U + \partial_x G(U) = 0$$

endowed with an entropy/entropy-flux pair $(\mathcal{E}, Q)$. We define the *relative entropy* $\mathcal{H}(U|\overline{U})$ as

$$\mathcal{H}(U|\overline{U}) := \mathcal{E}(U) - \mathcal{E}(\overline{U}) - D\mathcal{E}(\overline{U})(U - \overline{U}) \tag{10}$$

Note that this definition will only consider quadratic terms, but not linear terms in the entropy.
We choose this definition because if we have that $c_1, c_2 > 0$ and $D^2\mathcal{E}$ positive definite such that

$$c_1\mathbb{I} \leq D^2\mathcal{E} \leq c_2\mathbb{I},$$

then there are $c_3, c_4 > 0$ such that

$$c_3|U - \overline{U}|^2 \leq \mathcal{H}(U|\overline{U}) \leq c_4|U - \overline{U}|^2.$$

Weakly Dissipative Solutions I

Next, we define a stronger version of solution:

Definition (Weakly Dissipative Solutions)

$\{\varrho, \mathbf{u}, \eta\}$ are called a *weak dissipative solution* to the NSS system if and only if

- $\{\varrho, \mathbf{u}, \eta\}$ form a renormalized weak solution with the energy inequality

$$\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \eta\ln\eta + \eta\Phi \, \mathrm{d}x(\tau)$$
$$+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + |2\nabla_x\sqrt{\eta} + \sqrt{\eta}\nabla_x\Phi|^2 \, \mathrm{d}x \, \mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \eta_0\ln\eta_0 + \eta_0\Phi \, \mathrm{d}x$$
$$-\beta\int_0^\tau\int_\Omega \varrho\mathbf{u}\cdot\nabla_x\Phi \, \mathrm{d}x \, \mathrm{d}t \qquad (11)$$

  satisfied for all τ.

Weakly Dissipative Solutions II

Definition (Weakly Dissipative Solutions)

- for all suitably smooth solutions $\{r, \mathbf{U}, s\}$ of the NSS system, the following relative entropy inequality holds for all τ.

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s)\ \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\ \mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{U}_0|^2 + E_F(\varrho_0, r_0) + E_P(\eta_0, s_0)\ \mathrm{d}x \\
&+ \int_0^\tau \mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s)\ \mathrm{d}t
\end{aligned}
\tag{12}
$$

Remainder Term

The remainder term in (12) has the form

$$
\begin{aligned}
&\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s) \\
&:= \int_\Omega \mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{U})) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x - \int_\Omega \varrho(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x \\
&- \int_\Omega \partial_t P_F(r)(\varrho - r) + \nabla_x P_F(r) \cdot (\varrho \mathbf{u} - r\mathbf{U}) \ \mathrm{d}x \\
&- \int_\Omega [\varrho(P_F(\varrho) - P_F(r)) - E_F(\varrho, r)] \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \\
&- \int_\Omega \partial_t P_P(s)(\eta - s) + \nabla_x P_P(s) \cdot (\eta \mathbf{u} - s\mathbf{U}) \ \mathrm{d}x \\
&- \int_\Omega [\eta(P_P(\eta) - P_P(s)) - E_P(\eta, s)] \mathrm{div}_x \mathbf{U} \ \mathrm{d}x \\
&- \int_\Omega \nabla_x (P_P(\eta) - P_P(s)) \cdot (\nabla_x \eta + \eta \nabla_x \Phi) \ \mathrm{d}x \\
&- \int_\Omega \left[(\beta\varrho + \eta) \nabla_x \Phi + \frac{\eta \nabla_x s}{s} \right] \cdot (\mathbf{u} - \mathbf{U}) \ \mathrm{d}x
\end{aligned}
$$

Approximation Scheme

A three-level approximation scheme is employed

- Artificial pressure parameterized by small δ
- Vanishing viscosity parameterized by small ε
- Faedo-Galerkin approximation where test functions for the momentum equation are taken from n-dimensional function spaces X_n of smooth functions on $\overline{\Omega}$

Approximate System

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = \varepsilon \Delta_x \varrho_n \tag{13}$$

$$\partial_t \eta_n + \mathrm{div}_x(\eta_n \mathbf{u}_n - \eta_n \nabla_x \Phi) = \Delta_x \eta_n \tag{14}$$

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{w} \, \mathrm{d}x = \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x \mathbf{w} + (a\varrho_n^\gamma + \eta_n + \delta \varrho_n^\alpha) \mathrm{div}_x \mathbf{w} \, \mathrm{d}x$$
$$- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{w} + \varepsilon \nabla_x \varrho_n \cdot \nabla_x \mathbf{u}_n \cdot \mathbf{w} \, \mathrm{d}x - \int_\Omega (\beta \varrho_n + \eta_n) \nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x \tag{15}$$

with the additional conditions

$$\nabla_x \varrho_n \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial\Omega$$
$$\mathbf{u}_n = (\nabla_x \eta_n + \eta_n \nabla_x \Phi) \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial\Omega$$

Existence of Approximate Solutions

- Existence of $\mathbf{u}_n$ is obtained from the Faedo-Galerkin approximation and an iteration argument in the spirit of Feireisl.
- ϱ_n, η_n obtained from $\mathbf{u}_n$ using fixed point arguments in the spirit of Ladyzhenskaya.

Approximate Energy Inequality

Using $\mathbf{u}_n$ as a test function in (15) and some straight-forward manipulations:

$$\int_\Omega \frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \frac{a}{\gamma-1}\varrho_n^\gamma + \frac{\delta}{\alpha-1}\varrho_n^\alpha + \eta_n \ln \eta_n + \eta_n \Phi \ \mathrm{d}x(\tau)$$
$$+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n + |2\nabla_x \sqrt{\eta_n} + \sqrt{\eta_n}\nabla_x \Phi|^2 \ \mathrm{d}x \ \mathrm{d}t$$
$$+ \varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_n|^2 (a\gamma \varrho_n^{\gamma-2} + \delta a \varrho_n^{\alpha-2}) \ \mathrm{d}x \ \mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta}\ln \eta_{0,\delta} + \eta_{0,\delta}\Phi \ \mathrm{d}x$$
$$- \beta \int_0^\tau \int_\Omega \varrho_n \mathbf{u}_n \cdot \nabla_x \Phi \ \mathrm{d}x \ \mathrm{d}t \qquad (16)$$

Uniform Bounds

From the energy inequality, we find that

$$
\begin{aligned}
&\{\mathbf{u}\}_{n,\varepsilon,\delta} \in_b L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))\\
&\{\sqrt{\varrho}\mathbf{u}\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T; L^2(\Omega;\mathbb{R}^3))\\
&\{\varrho\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T; L^\gamma(\Omega))\\
&\{\eta \ln \eta\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T; L^1(\Omega))\\
&\{\nabla_x \sqrt{\eta}\}_{n,\varepsilon,\delta} \in_b L^2(0,T; L^2(\Omega;\mathbb{R}^3))\\
&\{\eta\}_{n,\varepsilon,\delta} \in_b L^2(0,T; W^{1,\frac{3}{2}}(\Omega))
\end{aligned}
$$

Faedo-Galerkin Limit I

- From the approximate energy balance, the term

$$\varepsilon\delta \int_0^T \int_\Omega |\nabla_x \varrho_n|^2 \varrho_n^{\alpha-2} \, \mathrm{d}x \, \mathrm{d}t$$

is bounded independently of n. Thus by Poincaré's inequality,

$$\{\varrho\}_n \in_b L^2(0, T; W^{1,2}(\Omega)).$$

- From this, $\nabla_x \varrho_n \cdot \mathbf{u}_n \in_b L^1(0, T; L^{3/2}(\Omega))$. To get higher time integrability, multiply (13) by $G'(\varrho_n)$ where $G(\varrho_n) := \varrho_n \ln \varrho_n$. Then

$$\varepsilon \int_0^T \int_\Omega \frac{|\nabla_x \varrho_n|^2}{\varrho_n} \, \mathrm{d}x \, \mathrm{d}t$$

is bounded independently of n. Using Hölder's and interpolation,

$$\{\nabla_x \varrho_n \cdot \mathbf{u}_n\}_n \in_b L^q(0, T; L^p(\Omega))$$

for some $p \in \left(1, \frac{3}{2}\right)$ and $q \in (1, 2)$. Thus, $\varrho_\varepsilon, \mathbf{u}_\varepsilon$ obey

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \Delta_x \varrho_\varepsilon.$$

Faedo-Galerkin Limit II

- Strong convergence of $\nabla_x \varrho_n \to \nabla_x \varrho_\varepsilon$ follows from letting $G(z) = z^2$.
- Similar techniques show convergence of $\eta_n \to \eta_\varepsilon$ and $\nabla_x \eta_n \to \nabla_x \eta_\varepsilon$ to allow
$$\partial_t \eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon - \eta_\varepsilon \nabla_x \Phi) = \Delta_x \eta_\varepsilon.$$
- Terms in the momentum equation converge as we want using the bounds and the above convergences, except for the convective term.
- Convergence of the convective term $\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n$ in $L^q((0,T) \times \Omega; \mathbb{R}^3)$ follows from convergence of $\varrho_n \mathbf{u}_n$ and Arzela-Ascoli.

The following lemma is of use throughout the analysis for convergence of the η terms:

Lemma (Simon)

Let $X \subset B \subset Y$ be Banach spaces with $X \subset B$ compactly. Then, for $1 \leq p < \infty$, $\{v : v \in L^p(0,T;X),\ v_t \in L^1(0,T;Y)\}$ is compactly embedded in $L^p(0,T;B)$.

Thus, $\{\eta\}_{n,\varepsilon} \to \eta_\delta$ in $L^2(0,T;L^3(\Omega))$.

Vanishing Viscosity Approximation I

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \Delta_x \varrho_\varepsilon \tag{17}$$

$$\partial_t \eta_\varepsilon + \mathrm{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon - \eta_\varepsilon \nabla_x \Phi) = \Delta_x \eta_\varepsilon \tag{18}$$

$$\begin{aligned}\int_\Omega \partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon)\cdot \mathbf{w}\ \mathrm{d}x &= \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w} + (a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta \varrho_\varepsilon^\alpha)\mathrm{div}_x \mathbf{w}\ \mathrm{d}x \\ &- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{w} + \varepsilon \nabla_x \varrho_\varepsilon \cdot \nabla_x \mathbf{u}_\varepsilon \cdot \mathbf{w}\ \mathrm{d}x - \int_\Omega (\beta \varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \cdot \mathbf{w}\ \mathrm{d}x\end{aligned} \tag{19}$$

$$\nabla_x \varrho_\varepsilon \cdot \mathbf{n} = 0$$

$$\mathbf{u}_\varepsilon|_{\partial\Omega} = (\nabla_x \eta_\varepsilon + \eta_\varepsilon \nabla_x \Phi)\cdot \mathbf{n}|_{\partial\Omega} = 0$$

Vanishing Viscosity Approximation II

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{\delta}{\alpha-1}\varrho_\varepsilon^\alpha + \eta_\varepsilon \ln \eta_\varepsilon + \eta_\varepsilon \Phi \, \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon + |2\nabla_x \sqrt{\eta_\varepsilon} + \sqrt{\eta_\varepsilon}\nabla_x \Phi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&+ \varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_\varepsilon|^2 (a\gamma \varrho_\varepsilon^{\gamma-2} + \delta a \varrho_\varepsilon^{\alpha-2}) \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta} \ln \eta_{0,\delta} + \eta_{0,\delta}\Phi \, \mathrm{d}x \\
&- \beta \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Phi \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}
\tag{20}
$$

Vanishing Viscosity Limit I

- We begin by using the uniform bounds and obtaining weak limits $\varrho_\delta, \mathbf{u}_\delta, \eta_\delta$.
- We show that since $\sqrt{\varepsilon}\nabla_x \varrho_\varepsilon \to 0$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$, and using Arzela-Ascoli, we have that $\varrho_\delta, \mathbf{u}_\delta$ solve the continuity equation weakly.
- Similar analysis shows that $\eta_\delta, \mathbf{u}_\delta$ solve the Smoluchowski equation weakly.

Vanishing Viscosity Limit II

- Convergence of the momentum equation is fairly straight-forward except for the pressure-related terms. Using the Bogovskii operator (analogous to an inverse divergence operator) and an appropriate test function, we find that $a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha$ has a weak limit.
- To show this weak limit is $a\varrho_\delta^\gamma + \eta_\delta + \delta\varrho_\delta^\alpha$, we have to show the strong convergence of the fluid density (strong convergence of the particle density follows from the lemma of Simon).
- This is obtained by using the test function $\psi(t)\zeta(x)\varphi_1(x)$ where $\psi \in C_c^\infty(0,T), \zeta \in C_c^\infty(\Omega), \varphi_1(x) := \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon)$, and analysis involving the double Reisz transform and the Div-Curl Lemma.

Artificial Pressure Approximation I

$$\int_0^T \int_\Omega \varrho_\delta B(\varrho_\delta)(\partial_t \phi + \mathbf{u}_\delta \cdot \nabla_x \phi) \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \varrho_{0,\delta} B(\varrho_{0,\delta}) \phi(0, \cdot) \, \mathrm{d}x$$

$$= \int_0^T \int_\Omega b(\varrho_\delta) \mathrm{div}_x \mathbf{u}_\delta \phi \, \mathrm{d}x \, \mathrm{d}t \tag{21}$$

$$\int_0^T \int_\Omega \eta_\delta \partial_t \phi + (\eta_\delta \mathbf{u}_\delta - \eta_\delta \nabla_x \Phi - \nabla_x \eta_\delta) \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega \eta_{0,\delta} \phi(0, \cdot) \, \mathrm{d}x \tag{22}$$

$$\int_\Omega \partial_t (\varrho_\delta \mathbf{u}_\delta) \mathbf{w} \, \mathrm{d}x = \int_\Omega \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla_x \mathbf{w} + (a \varrho_\delta^\gamma + \eta_\delta + \delta \varrho_\delta^\alpha) \mathrm{div}_x \mathbf{w} \, \mathrm{d}x$$

$$- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{w} \, \mathrm{d}x - \int_\Omega (\beta \varrho_\delta + \eta_\delta) \nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x \tag{23}$$

Artificial Pressure Approximation II

$$\int_\Omega \frac{1}{2}\varrho_\delta |\mathbf{u}_\delta|^2 + \frac{a}{\gamma-1}\varrho_\delta^\gamma + \frac{\delta}{\alpha-1}\varrho_\delta^\alpha + \eta_\delta \ln \eta_\delta + \eta_\delta \Phi \, \mathrm{d}x(\tau)$$
$$+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{u}_\delta + |2\nabla_x \sqrt{\eta_\delta} + \sqrt{\eta_\delta}\nabla_x \Phi|^2 \, \mathrm{d}x \, \mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_{0,\delta} |\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta} \ln \eta_{0,\delta} + \eta_{0,\delta} \Phi \, \mathrm{d}x$$
$$- \beta \int_0^\tau \int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \nabla_x \Phi \, \mathrm{d}x \, \mathrm{d}t \tag{24}$$

Artificial Pressure Limit

- Again, from uniform bounds, we are able to obtain the existence of weak limits $\varrho, \mathbf{u}, \eta$.
- Much of the difficulty in taking the artificial pressure limit is controlling the oscillation defect measure for the fluid density ϱ.

Oscillation Defect Measure and Strong Convergence

Definition

Let $Q \subset \Omega$ and $q \geq 1$. Then

$$\mathbf{osc}_q[\varrho_\delta - \varrho](Q) := \sup_{k \geq 1} \left(\limsup_{\delta \to 0^+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}x \right).$$

Here, $\{T_k\}$ is a family of appropriately concave cutoff functions. Using these cutoff functions, we can control the oscillation defect measure and obtain strong convergence of the fluid density.

Approximate Relative Entropy Inequality

- We formulate an approximate relative entropy inequality for each fixed n, ε, δ.
- We define smooth functions $\mathbf{U}_m \in C^1([0, T]; X_m)$ zero on the boundary and positive r_m, s_m on $[0, T] \times \overline{\Omega}$.
- We take $\mathbf{u}_n - \mathbf{U}_m$ as a test function on the Faedo-Galerkin approximate momentum equation and perform some calculations to obtain an approximate relative entropy inequality.
- We take the limits to obtain the relative entropy inequality.

Relative Entropy Inequality

Regularity of $r, \mathbf{U}, s$ are imposed to ensure that all integrals in the formula for the relative entropy are defined.

$$
\begin{aligned}
& r \in C_{\text{weak}}([0,T]; L^{\gamma}(\Omega)) \\
& \mathbf{U} \in C_{\text{weak}}([0,T]; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \\
& \nabla_x \mathbf{U} \in L^2(0,T; L^2(\Omega;\mathbb{R}^{3\times 3})),\ \mathbf{U}|_{\partial\Omega} = 0 \\
& s \in C_{\text{weak}}([0,T]; L^1(\Omega)) \cap L^1(0,T; L^{6\gamma/\gamma-3}(\Omega)) \\
& \partial_t \mathbf{U} \in L^1(0,T; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3)) \\
& \nabla_x^2 \mathbf{U} \in L^1(0,T; L^{2\gamma/\gamma+1}(\Omega;\mathbb{R}^{3\times 3\times 3}) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^{3\times 3\times 3})) \\
& \partial_t P_F(r) \in L^1(0,T; L^{\gamma/\gamma-1}(\Omega)) \\
& \nabla_x P_F(r) \in L^1(0,T; L^{2\gamma/\gamma-1}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega;\mathbb{R}^3)) \\
& \partial_t P_P(s) \in L^1(0,T; L^{\infty}(\Omega)) \cap L^{\infty}(0,T; L^{3/2}(\Omega)) \\
& \nabla_x P_P(s) \in L^{\infty}(0,T; L^3(\Omega;\mathbb{R}^3)) \\
& \nabla_x s \in L^{\infty}(0,T; L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma+3}(\Omega;\mathbb{R}^3)).
\end{aligned}
\tag{25}
$$

Uniqueness of Weakly Dissipative Solutions

Theorem (Weak-Strong Uniqueness)

Assume $\{\varrho, \mathbf{u}, \eta\}$ *is a weakly dissipative solution of the NSS system. Assume that* $\{r, \mathbf{U}, s\}$ *is a smooth solution of the NSS system with appropriate regularity with the same initial data. Then* $\{\varrho, \mathbf{u}, \eta\}$ *and* $\{r, \mathbf{U}, s\}$ *are identical.*

Note that the following hypotheses are imposed on the smooth solutions

$$\begin{aligned}
&\nabla_x r \in L^2(0, T; L^q(\Omega; \mathbb{R}^3)) \\
&\nabla_x^2 \mathbf{U} \in L^2(0, T; L^q(\Omega; \mathbb{R}^{3\times3\times3})) \\
&\alpha := \nabla_x s + s\nabla_x \Phi \in L^2(0, T; L^q(\Omega; \mathbb{R}^3))
\end{aligned} \tag{26}$$

where

$$q > \max\left\{3, \frac{3}{\gamma - 1}\right\}$$

The proof involves analysis bounding the remainder terms in terms of the relative entropy and using Gronwall's inequality on the result.

Remarks

- The result can be generalized to unbounded spatial domains by creating a sequence of bounded domains and passing the limits through using the confinement hypotheses.
- This result does not show the existence of appropriately smooth $\{r, \mathbf{U}, s\}$, which is the focus of current work.
  - First, existence of local strong solutions for appropriate initial data will be shown along the lines of Cho and Kim.
  - Second, appropriate blow-up conditions will be formulated that will enable us to extend the local result to a global result in the style of Fan, Jian, and Ou.

Local Existence of Strong Solutions

Proof for local-in-time existence requires the following regularity on the initial data with $q \in (3, 6]$

$$\begin{aligned} &\varrho_0 \in W^{1,q}(\Omega) \\ &\mathbf{u}_0 \in W_0^{1,2}(\Omega; \mathbb{R}^3) \cap W^{2,2}(\Omega; \mathbb{R}^3) \\ &\eta_0 \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \end{aligned} \tag{27}$$

and a vector field $\mathbf{h} \in L^2(\Omega; \mathbb{R}^3)$ satisfying the compatibility conditions

$$\begin{aligned} &\Phi \in W^{2,2}(\Omega) \\ &\sqrt{\varrho_0}\mathbf{h} = \nabla_x(a\varrho_0^\gamma + \eta_0) - \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_0) + \eta_0 \nabla_x \Phi \end{aligned} \tag{28}$$

Local Existence Result

Theorem (Local In Time Existence)

Consider the NSS system (1)-(3) *with the boundary conditions* (5), *initial conditions* (27) *and compatibility conditions* (28). *Then there exists a unique solution* $\{\varrho, \mathbf{u}, \eta\}$ *such that*

$$
\begin{aligned}
&\varrho \in C([0,T]; W^{1,q}(\Omega)) \\
&\varrho_t \in C([0,T]; L^q(\Omega)) \\
&\mathbf{u} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3)) \\
&\mathbf{u}_t \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)) \\
&\eta \in C([0,T]; W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0,T; W^{2,q}(\Omega)) \\
&\eta_t \in L^2(0,T; W_0^{1,2}(\Omega)).
\end{aligned}
$$

for some finite $T > 0$.

Linear System

Analysis for local existence of strong solutions uses existence and estimates on solutions to the *linear NSS system*

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{v}) = 0 \tag{29}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{v} \otimes \mathbf{u}) + \nabla_x(a\varrho^\gamma + \eta)$$
$$= \mu \Delta_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u} - (\beta \varrho + \eta)\nabla_x \Phi \tag{30}$$

$$\partial_t \eta + \mathrm{div}_x(\eta \mathbf{v} - \eta \nabla_x \Phi) - \Delta_x \eta = 0 \tag{31}$$

where

$$\mathbf{v} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3))$$

$$\mathbf{v}_t \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)).$$

We also assume

$$0 < \delta \leq \varrho_0$$

to approximate the initial fluid density with one that does not have a vacuum.

Linear Approximation Existence

Using the method of characteristics and classical results on parabolic equations, we obtain the existence of solutions $\{\varrho, \mathbf{u}, \eta\}$ to (29)-(31) such that for some $T > 0$,

$$\varrho \in C([0,T]; W^{1,q}(\Omega)),\ \varrho_t \in C([0,T]; L^q(\Omega))$$
$$\eta \in C([0,T]; W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0,T; W^{2,q}(\Omega))$$
$$\eta_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega))$$
$$\eta_{tt} \in L^2(0,T; W^{-1,2}(\Omega))$$
$$\mathbf{u} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3))$$
$$\mathbf{u}_t \in C([0,T]; L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$$
$$\mathbf{u}_{tt} \in L^2(0,T; W^{-1,2}(\Omega;\mathbb{R}^3)).$$

Uniform Bounds

In order to be able to pass through the limit of $\delta \to 0$, we obtain the bounds uniform in δ on various quantities. Key among them are the ones on fluid density ϱ and pressure $a\varrho^\gamma + \eta$ below.

$$
\begin{aligned}
&\|\varrho(t)\|_{W^{1,q}(\Omega)} \leq Cc_0 \\
&\|\varrho_t(t)\|_{L^q(\Omega)} \leq Cc_2 \\
&P(\varrho, \eta)(t) \text{ is continuous on } \Omega \\
&\|\nabla_x P(\varrho, \eta)(t)\|_{L^q(\Omega;\mathbb{R}^3)} \leq Cc_0 + c_g \\
&\|\partial_t P(\varrho, \eta)(t)\|_{L^2(\Omega)} \leq Cc_2 + c_g
\end{aligned}
$$

The pressure bounds are used to obtain δ-independent bounds on $\mathbf{u}$, and then the analysis follows that of Cho and Kim.

Vacuum Case

Because of the uniform bounds, we can take $\delta \to 0$, eliminating the positive lower bound for ϱ_0.

- For each δ, a positive initial density $\varrho_0^\delta := \varrho_0 + \delta$ and an approximation for $\mathbf{h}$, $\mathbf{h}^\delta$ is defined. Because of the uniform-in-δ bounds, we find a solution $\{\varrho, \mathbf{u}, \eta\}$ to the linear problem.
- These solutions are shown to converge to a solution of the linear problem with the following regularity.

Regularity of Linear-Vacuum System Solutions

$$
\begin{aligned}
&\varrho \in C([0,T]; W^{1,q}(\Omega)),\ \varrho_t \in C([0,T]; L^q(\Omega)) \\
&\eta \in C([0,T]; W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0,T; W^{2,q}(\Omega)) \\
&\eta_t \in L^2(0,T; W_0^{1,2}(\Omega)) \\
&\mathbf{u} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3)) \\
&\eta_t \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)) \\
&\sqrt{\varrho}\mathbf{u}_t \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)).
\end{aligned} \tag{32}
$$

Solutions to Nonlinear System

To find solutions to the nonlinear system, we find a sequence of solutions to the linear system with a sequence of functions $\{\mathbf{v}^k\}$ defined inductively.

- $\mathbf{v}^0$ solves the initial value problem

$$\partial_t \mathbf{w} - \Delta_x \mathbf{w} = 0,\ \mathbf{w}(0, \cdot) = \mathbf{u}_0$$

- $\mathbf{v}^{k+1} = \mathbf{u}^k$ where $\mathbf{u}^k$ solves the linear system using $\mathbf{v}^k$ for $\mathbf{v}$.
- Using this induction, we get a sequence of solutions $\{\varrho^k, \mathbf{u}^k, \eta^k\}$ that converge to some $\{\varrho, \mathbf{u}, \eta\}$ which are smooth and solve the nonlinear system (1)-(3) for some finite time $T > 0$.
- Combined with the weak-strong uniqueness result, we know that if the initial data have compatibility conditions (27)-(28), then there is a unique smooth solution for some finite time.

Conclusion

- We have given another proof of the existence of renormalized solutions to the NSS system, albeit with a slightly different energy inequality. The key aspect of this class of solution is that it obeys the relative entropy inequality.
- If there is a solution with regularity given in (25)-(26), then there is only one weakly dissipative solution.
- If the initial data satisfy (27) and (28), then there is a unique strong solution for finite time.
- It remains to develop blow-up conditions for the NSS system.

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