Presentation Problems 5

21-355 A

For these problems, assume all sets are subsets of $\mathbb{R}$ unless otherwise specified.

1. Let P and Q be partitions of $[a,b]$ such that $P \subseteq Q$. Then $U(f,P) \geq U(f,Q)$ and $L(f,P) \leq L(f,Q)$. Use this to show that for any partitions P_1 and P_2 of $[a,b]$ that $L(f,P_1) \leq U(f,P_2)$.

   Proof. First, we will prove that $U(f,P) \geq U(f,Q)$ and $L(f,P) \leq L(f,Q)$. Let $[x_{k-1}, x_k]$ be an subinterval of P and suppose there exists $z \in Q$ such that $x_{k-1} < z < x_k$. Denote

$$\begin{cases} M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\} \\ M_k' = \sup\{f(x) : x \in [x_{k-1}, z]\} \\ M_k'' = \sup\{f(x) : x \in [z, x_k]\} \end{cases}$$

   It follows that $M_k \geq M_k'$ and $M_k \geq M_k''$, and we have that

$$\begin{aligned} M_k\Delta_k &= M_k(x_k - x_{k-1}) \\ &= M_k(x_k - z + z - x_{k-1}) \\ &= M_k(x_k - z) + M_k(z - x_{k-1}) \\ &\geq M_k''(x_k - z) + M_k'(z - x_{k-1}) \end{aligned}$$

   Further, we can employ induction to prove this fact for any finite number of points in this interval $[x_{k-1}, x_k]$ that are also in Q. As such, we know that the upper sum cannot get larger when we add more points to a partition. That is $U(f,P) \geq U(f,Q)$.

   Similarly, denote

$$\begin{cases} m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\} \\ m_k' = \inf\{f(x) : x \in [x_{k-1}, z]\} \\ m_k'' = \inf\{f(x) : x \in [z, x_k]\} \end{cases}$$

It follows that $m_k \leq m'_k$ and $m_k \leq m''_k$, and we have that

$$\begin{aligned} m_k\Delta_k &= m_k(x_k - x_{k-1}) \\ &= m_k(x_k - z + z - x_{k-1}) \\ &= m_k(x_k - z) + m_k(z - x_{k-1}) \\ &\leq m''_k(x_k - z) + m'_k(z - x_{k-1}) \end{aligned}$$

Then the lower sum cannot get smaller when we add more points to a partition. That is $L(f, P) \leq L(f, Q)$.

Consider partitions P_1 and P_2 of $[a, b]$. Define $Q = P_1 \cup P_2$. Then $P_1 \subseteq Q$ and $P_2 \subseteq Q$. Then by our previously proved fact

$$L(f, P_1) \leq L(f, Q) \leq U(f, Q) \leq U(f, P_2).$$

□

2. Let $f : [a, b] \mapsto \mathbb{R}$ be bounded. Then f is integrable on $[a, b]$ if and only if for all $\varepsilon > 0$, there exists some partition P_ε of $[a, b]$ such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Proof. ($\Leftarrow$) Let $\epsilon > 0$. Then by assumption, there exists some partition P_ϵ such that

$$U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon.$$

However, $U(f) \leq U(f, P_\epsilon)$. $L(f) \geq L(f, P_\epsilon)$. Then, $U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$. Thus, $0 \leq U(f) - L(f) < \epsilon$ for all $\epsilon > 0$, so $U(f) = L(f)$, and thus f is integrable.

($\Rightarrow$) Since $U(f)$ is the infimum of all the upper sums, for $\epsilon > 0$, we can find P_1 such that $U(f, P_1) < U(f) + \frac{\epsilon}{2}$. Similarly, we can find P_2 such that $L(f, P_2) > L(f) - \frac{\epsilon}{2}$. Now, we can define $P_\epsilon = P_1 \cup P_2$. Then,

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) &\leq U(f, P_1) - L(f, P_2) \\ &< U(f) + \frac{\epsilon}{2} - (L(f) - \frac{\epsilon}{2}) \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ \text{because } f \text{ is integrable, } U(f) = L(f) \\ &= \epsilon \end{aligned}$$

□

3. Let $f : [a, b] \mapsto \mathbb{R}$ be bounded. Then f is integrable on $[a, b]$ if and only if there exists some sequence of partitions (P_n) of $[a, b]$ such that

$$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0.$$

Proof. $(\Rightarrow)$ Let $\epsilon > 0, \exists$ some natural number N such that $U(f, P_N) - L(f, P_N) < \epsilon$ It follows from the result of problem number 2 that f is integrable

$(\Leftarrow)$ Assume f is integrable so P_n is a sequence of p guaranteed that $U(f, P_n) - L(f, P_n) < 1/n$ then it follows from the result of problem number 2 that there exists P_n such that
$\lim_{n\to\infty} U(f, P_n) - L(f, P_n) = 0$

□

4. Let $f : [a, b] \mapsto \mathbb{R}$ be increasing. Then f is integrable on $[a, b]$.

Proof. Since $f : [a, b] \to \mathbb{R}$ is an increasing function, $f(a) \leq f(x) \leq f(b)$ for all $x \in [a, b]$. Therefore, f is bounded on $[a, b]$.

Let $d = \frac{b-a}{n}$. Let P_n be a finite set of points $\{x_0, x_1, x_2, ..., x_n\}$ such that $x_k = a + kd$. Then P_n is a partition of $[a, b]$ because $a = x_0 < a + d = x_1 < a + 2d = x_2 < \ldots < a + nd = x_n = b$. Since f is increasing, we have

$$m_k = \inf_{x\in[x_{k-1},x_k]} f(x) = f(x_{k-1}).$$
$$M_k = \sup_{x\in[x_{k-1},x_k]} f(x) = f(x_k).$$

Therefore,

$$U(f, P_n) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \frac{b-a}{n} \sum_{k=1}^{n} f(x_k)$$
$$L(f, P_n) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \frac{b-a}{n} \sum_{k=1}^{n} f(x_{k-1})$$

And

$$\begin{aligned}
\lim_{x\to\infty} U(f,P_n) - L(f,P_n) &= \lim_{x\to\infty} \Big(\frac{b-a}{n} \sum_{k=1}^{n} f(x_k) - \frac{b-a}{n} \sum_{k=1}^{n} f(x_{k-1}) \Big) \\
&= \lim_{x\to\infty} \Big(\frac{b-a}{n} \Big(\sum_{k=1}^{n} f(x_k) - \sum_{k=1}^{n} f(x_{k-1}) \Big) \Big) \\
&= \lim_{x\to\infty} \Big(\frac{b-a}{n} \Big(\sum_{k=1}^{n} f(x_k) - \sum_{k=0}^{n-1} f(x_k) \Big) \Big) \\
&= \lim_{x\to\infty} \Big(\frac{b-a}{n} (f(x_n) - f(x_0)) \Big) \\
&= \lim_{x\to\infty} \Big(\frac{(b-a)(f(b)-f(a))}{n} \Big) \\
&= 0
\end{aligned}$$

Therefore, by Presentation 5 Problem 3, we have f is integrable on $[a,b]$.
□

5. Let $f : [a,b] \mapsto \mathbb{R}$ be continuous. Then f is integrable on $[a,b]$.

   Proof. Let $f : [a,b] \mapsto \mathbb{R}$ be continuous
   Notice that $[a,b]$ is compact, therefore f is uniformly continuous.
   Arbitrarily pick $\epsilon > 0$
   $\exists \delta > 0$ such that $|x-y| < \delta \implies |f(x) - f(y)| < \epsilon/(b-a)$
   Now let P be a partition of $[a,b]$ where the distance between any consecutive term is smaller than δ
   Given a particular subinterval $[x_{k-1}, x_k]$ of the P
   Based on the extreme value theorem, the function achieves extreme value somewhere in the bound.
   Hence $M_k = f(z_k)$ for some $z_k \in [x_{k-1}, x_k]$ and also $m_k = f(y_k)$ for some $y_k \in [x_{k-1}, x_k]$
   Since $|x_k - x_{k-1}| < \delta$ and $z_k, y_k \in [x_{k-1}, x_k]$
   $|z_k - y_k| < \delta$
   So $M_k - m_k = f(z_k) - f(y_k) < \epsilon/(b-a)$
   $U(f,P) - L(f,P) = \sum_{k=1}^{n} (M_k - m_k)(x_k - x_{k-1})$
   $< \sum_{k=1}^{n} \epsilon/(b-a) * (x_k - x_{k-1})$
   $= \epsilon/(b-a) * (b-a)$
   $= \epsilon$
   Hence $U(f,P) - L(f,P) < \epsilon$
   Therefore f is integrable. □

6. Let f, g be integrable on $[a,b]$ and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is integrable on $[a,b]$ and

$$\int_a^b \alpha f + \beta g \, \mathrm{d}x = \alpha \int_a^b f \, \mathrm{d}x + \beta \int_a^b g \, \mathrm{d}x.$$

Proof. Since f,g are integrable, by presentation problem 3, we know there exists sequence of partitions $(P_n), (Q_n)$ s.t.

$$\lim_{n\to\infty}[U(f, P_n) - L(f, P_n)] = 0 \;\; \& \;\; \lim_{n\to\infty}[U(g, Q_n) - L(g, Q_n)] = 0$$

Then, we can create a sequence of partitions $(K_n) = (P_n \cup Q_n)$. Since $P_n \subseteq K_n$ and $Q_n \subseteq K_n$, we know (by problem 1):

- $L(f, P_n) \leq L(f, K_n) \leq U(f, K_n) \leq U(f, P_n)$
- $L(g, Q_n) \leq L(g, K_n) \leq U(g, K_n) \leq U(g, Q_n)$

which means:

$U(f, K_n) - L(f, K_n) \leq U(f, P_n) - L(f, P_n)$ and $U(g, K_n) - L(g, K_n) \leq U(g, Q_n) - L(g, Q_n)$.

Because $U(f, K_n) - L(f, K_n) \geq 0$ while it is less than or equal to $U(f, P_n) - L(f, P_n)$, the $\lim_{n\to\infty}[U(f, K_n) - L(f, K_n)]$ is squeezed to 0; it is the same case for $\lim_{n\to\infty}[U(g, K_n) - L(g, K_n)] = 0$.

Using Algebratic Limit Thm, we obtain $\lim_{n\to\infty} U(f, K_n) = \lim_{n\to\infty} L(f, K_n)$ and $\lim_{n\to\infty} U(g, K_n) = \lim_{n\to\infty} L(g, K_n)$.

Lemma: $U(\alpha f + \beta g, P) = \alpha X(f, P) + \beta Y(g, P)$, where $X = U$ if $\alpha > 0$, $X = L$ otherwise, and $Y = U$ if $\beta > 0$, $Y = L$ otherwise. And vice verse for $L(\alpha f + \beta g, P)$.

Proof: We know that if $A \subseteq \mathbb{R}$ and $cA := \{c \cdot a \mid a \in A\}$, then if $c \geq 0$, $\sup cA = c \sup A$ and $\inf cA = c \inf A$, but if $c < 0$ then $\sup cA = c \inf A$ and $\inf cA = c \sup A$.

So if $f(x) = a \cdot g(x)$ for some $a \geq 0 \in \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$, then

$$m_k = \inf\{f(x) \mid x \in [x_{k-1}, x_k]\} = \inf\{a \cdot g(x) \mid x \in [x_{k-1}, x_k]\} = a \inf\{g(x) \mid x \in [x_{k-1}, x_k]\}$$
$$M_k = \sup\{f(x) \mid x \in [x_{k-1}, x_k]\} = \sup\{a \cdot g(x) \mid x \in [x_{k-1}, x_k]\} = a \sup\{g(x) \mid x \in [x_{k-1}, x_k]\}$$

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} a \inf\{g(x) \mid x \in [x_{k-1}, x_k]\}(x_k - x_{k-1}) = aL(g, P)$$
$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} a \sup\{g(x) \mid x \in [x_{k-1}, x_k]\}(x_k - x_{k-1}) = aU(g, P)$$

And if $f(x) = a \cdot g(x)$ for some $a < 0 \in \mathbb{R}$ and $g : \mathbb{R} \mapsto \mathbb{R}$, then

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} a \sup\{g(x) \mid x \in [x_{k-1}, x_k]\}(x_k - x_{k-1}) = aU(g, P)$$
$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} a \inf\{g(x) \mid x \in [x_{k-1}, x_k]\}(x_k - x_{k-1}) = aL(g, P)$$

So we have scalar multiplication of L and U, with the property we are trying to show. Also on a homework problem, we went over that suprema add, which can be used to say that infima add as well. Thus, U and L satisfy the claim. Now we will resume the proof. $\square$.

$$\lim_{n\to\infty}[U(\alpha f+\beta g,K_n)-L(\alpha f+\beta g,K_n)]$$
$$=\lim_{n\to\infty}U(\alpha f+\beta g,K_n)-\lim_{n\to\infty}L(\alpha f+\beta g,K_n)$$
$$=\lim_{n\to\infty}U(\alpha f,K_n)+\lim_{n\to\infty}U(\beta g,K_n)-\lim_{n\to\infty}L(\alpha f,K_n)-\lim_{n\to\infty}L(\beta g,K_n)$$

Now we will case on α and β being greater than or equal to zero.

If $\alpha \geq 0$ and $\beta < 0$ then

$$\lim_{n\to\infty}[U(\alpha f+\beta g,K_n)-L(\alpha f+\beta g,K_n)]$$
$$=\lim_{n\to\infty}U(\alpha f,K_n)+\lim_{n\to\infty}U(\beta g,K_n)-\lim_{n\to\infty}L(\alpha f,K_n)-\lim_{n\to\infty}L(\beta g,K_n)$$
$$=\alpha\lim_{n\to\infty}U(f,K_n)+\beta\lim_{n\to\infty}L(g,K_n)-\alpha\lim_{n\to\infty}U(f,K_n)-\beta\lim_{n\to\infty}L(g,K_n)=0$$

Similarly, if $\alpha < 0$ and $\beta \geq 0$, then

$$\lim_{n\to\infty}[U(\alpha f+\beta g,K_n)-L(\alpha f+\beta g,K_n)]$$
$$=\lim_{n\to\infty}U(\alpha f,K_n)+\lim_{n\to\infty}U(\beta g,K_n)-\lim_{n\to\infty}L(\alpha f,K_n)-\lim_{n\to\infty}L(\beta g,K_n)$$
$$=\alpha\lim_{n\to\infty}L(f,K_n)+\beta\lim_{n\to\infty}U(g,K_n)-\alpha\lim_{n\to\infty}L(f,K_n)-\beta\lim_{n\to\infty}U(g,K_n)=0$$

If both are less than 0, then

$$\lim_{n\to\infty}[U(\alpha f+\beta g,K_n)-L(\alpha f+\beta g,K_n)]$$
$$=\lim_{n\to\infty}U(\alpha f,K_n)+\lim_{n\to\infty}U(\beta g,K_n)-\lim_{n\to\infty}L(\alpha f,K_n)-\lim_{n\to\infty}L(\beta g,K_n)$$
$$=\alpha\lim_{n\to\infty}L(f,K_n)+\beta\lim_{n\to\infty}L(g,K_n)-\alpha\lim_{n\to\infty}U(f,K_n)-\beta\lim_{n\to\infty}U(g,K_n)=0$$
$$=-\alpha(\lim_{n\to\infty}U(f,K_n)-\lim_{n\to\infty}L(f,K_n))-\beta(\lim_{n\to\infty}U(g,K_n)-\lim_{n\to\infty}L(g,K_n))$$
$$=-\alpha\lim_{n\to\infty}[U(f,K_n)-L(f,K_n)]-\beta\lim_{n\to\infty}[U(g,K_n)-L(g,K_n)]$$
$$=-\alpha\cdot 0-\beta\cdot 0=0$$

If both are greater than 0, then

$$\lim_{n\to\infty}[U(\alpha f+\beta g,K_n)-L(\alpha f+\beta g,K_n)]$$
$$=\lim_{n\to\infty}U(\alpha f,K_n)+\lim_{n\to\infty}U(\beta g,K_n)-\lim_{n\to\infty}L(\alpha f,K_n)-\lim_{n\to\infty}L(\beta g,K_n)$$
$$=\alpha\lim_{n\to\infty}U(f,K_n)+\beta\lim_{n\to\infty}U(g,K_n)-\alpha\lim_{n\to\infty}L(f,K_n)-\beta\lim_{n\to\infty}L(g,K_n)=0$$
$$=\alpha(\lim_{n\to\infty}U(f,K_n)-\lim_{n\to\infty}L(f,K_n))+\beta(\lim_{n\to\infty}U(g,K_n)-\lim_{n\to\infty}L(g,K_n))$$
$$=\alpha\lim_{n\to\infty}[U(f,K_n)-L(f,K_n)]+\beta\lim_{n\to\infty}[U(g,K_n)-L(g,K_n)]$$
$$=\alpha\cdot 0+\beta\cdot 0=0$$

Since in all cases the limit is zero, we have that $\alpha f + \beta g$ is integrable.
Since

$$\int_a^b f\ dx = \lim_{n\to\infty} U(f, K_n)$$

and

$$\int_a^b g\ dx = \lim_{n\to\infty} U(g, K_n)$$

Given $\alpha f + \beta g$ is integrable, we know that $\int_a^b f dx = L(f, K_n) = U(f, K_n)$ and $\int_a^b g dx = L(g, K_n) = U(g, K_n)$. Thus, we can say

$$\begin{aligned}
\int_a^b \alpha f + \beta g\ dx &= \lim_{n\to\infty} U(\alpha f + \beta g, K_n) \\
&= \alpha \lim_{n\to\infty} X(f, K_n) + \beta \lim_{n\to\infty} Y(g, K_n) \\
&= \alpha \int_a^b f\ dx + \beta \int_a^b g\ dx \qquad \textit{since } X,\ Y \textit{ are either } U \textit{ or } L
\end{aligned}$$

□

7. Let f, g be integrable on $[a, b]$

   (a) If $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

   $$m(b-a) \leq \int_a^b f\ \mathrm{d}x \leq M(b-a).$$

   Proof. Recall that

   $$L(f) = \sup\{L(f, P) : P \in \mathcal{P}([a, b])\}$$

   $$U(f) = \inf\{U(f, P) : P \in \mathcal{P}([a, b])\}$$

   Let $P \in \mathcal{P}([a, b])$ where $|P| = n$.
   Then by definition of supremum and infimum, we have

   $$L(f, P) \leq L(f) \leq U(f) \leq U(f, P)$$

   By definition, we know that

   $$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1})$$

   $$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1})$$

where

$$m_k = \inf \{f(x) : x \in [x_{k-1}, x_k]\}$$
$$M_k = \sup \{f(x) : x \in [x_{k-1}, x_k]\}$$

Since M, m are upper and lower bounds on f, we have that

$$m \leq m_k$$
$$M_k \leq M$$

Then we can see that

$$\begin{aligned} L(f, P) &= \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \\ &\geq m \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= m(x_n - x_0) \\ &= m(b - a) \end{aligned}$$

Similarly, we have that

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \\ &\leq M \sum_{k=1}^{n} (x_k - x_{k-1}) \\ &= M(x_n - x_0) \\ &= M(b - a) \end{aligned}$$

By definition of integrability, we know that

$$L(f) = \int_a^b f \; dx = U(f)$$

It follows that

$$m(b - a) \leq \int_a^b f \; dx \leq M(b - a)$$

□

(b) If $f(x) \leq g(x)$ on $[a, b]$, then

$$\int_a^b f \; \mathrm{d}x \leq \int_a^b g \; \mathrm{d}x.$$

Proof. Since $f(x) \leq g(x)$, we know that

$$h(x) = f(x) - g(x) \leq 0$$

Then by part (a), we know that

$$\int_a^b h = \int_a^b (f-g)\ dx \leq 0 \cdot (b-a) = 0$$

By linearity, we know that

$$\int_a^b (f-g)\ dx = \int_a^b f\ dx - \int_a^b g\ dx$$

It follows that

$$\int_a^b f\ dx - \int_a^b g\ dx \leq 0$$

so

$$\int_a^b f\ dx \leq \int_a^b g\ dx$$

□

(c) $|f|$ is integrable on $[a, b]$ and

$$\left| \int_a^b f\ \mathrm{d}x \right| \leq \int_a^b |f|\ \mathrm{d}x.$$

Proof. First, we prove that $|f|$ is integrable Let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of $[a, b]$.
Claim: We show that for any interval $I_k = [x_{k-1}, x_k]$,

$$\sup_{I_k} |f| - \inf_{I_k} |f| \leq \sup_{I_k} f - \inf_{I_k} f$$

Proof: By the triangle inequality, for any $x, y \in I_k$,

$$|f(x)| - |f(y)| \leq |f(x) - f(y)|$$

Since

$$|f(x)| - |f(y)| \leq |f(x) - f(y)| = \max\{f(x), f(y)\} - \min\{f(x), f(y)\} \leq \sup_{I_k} f - \inf_{I_k} f$$

then $\sup_{I_k} f - \inf_{I_k} f$ is an upper bound on $|f(x)| - |f(y)|$ and so

$$\sup\{|f(x)| - |f(y)| : x, y \in I_k\} \leq \sup_{I_k} f - \inf_{I_k} f$$

Since

$$\sup\{|f(x)| - |f(y)| : x, y \in I_k\} = \sup_{I_k} |f| - \inf_{I_k} |f|$$

the claim holds.
So we have, for all $k = 1, \dots, n$,

$$\sup_{I_k} |f| - \inf_{I_k} |f| \leq \sup_{I_k} f - \inf_{I_k} f$$

$$\sum_{k=1}^{n} (\sup_{I_k} |f| - \inf_{I_k} |f|)(x_k - x_{k-1}) \leq \sum_{k=1}^{n} (\sup_{I_k} f - \inf_{I_k} f)(x_k - x_{k-1})$$

$$U(|f|, P) - L(|f|, P) \leq U(f, P) - L(f, P)$$

By the theorem from class, a function g is integrable on $[a, b]$ if and only if for all $\epsilon > 0$, there exists $P_\epsilon \in \mathcal{P}$ such that $U(g, P_\epsilon) - L(g, P_\epsilon) < \epsilon$.
We are given that it holds for f. For arbitrary $\epsilon > 0$, we know that $U(|f|, P_\epsilon) - L(|f|, P_\epsilon) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$, so it also holds for $|f|$. Therefore, $|f|$ is integrable on $[a, b]$.
Next, since $|f|$ is integrable,

$$-|f(x)| \leq f(x) \leq |f(x)|$$

Then by part (b), we know that

$$\int_a^b -|f| \, dx \leq \int_a^b f \, dx \leq \int_a^b |f| \, dx$$

By linearity, we have

$$-\int_a^b |f| \, dx \leq \int_a^b f \, dx \leq \int_a^b |f| \, dx$$

It follows that

$$\left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx$$

□

8. Let (f_n) be a sequence of real-valued functions on $[a, b]$ integrable on $[a, b]$. If $f_n \to f$ uniformly on $[a, b]$, then f is integrable on $[a, b]$ and

$$\lim_{n \to \infty} \int_a^b f_n \, \mathrm{d}x = \int_a^b f \, \mathrm{d}x.$$

Proof. We will begin by showing that f is bounded on $[a, b]$. Since $f_n \to f$ uniformly, for any $\epsilon > 0$ there exists an $N \in \mathbb{N}^+$ such that for all $n > N$, $x \in [a, b]$, we have $|f_n(x) - f(x)| < \epsilon$. Let $\epsilon = 1$ be arbitrary and choose an $M > N$ that is guaranteed by our assertion. f_n is integrable for any function in our sequence so we also know it is bounded. Then let us choose a $B > 0$ that is guaranteed by this property, giving us $|f(x)| < B$ for all $x \in [a, b]$. Then by the Triangle Inequality,

$$|f(x)| = |f(x) - f_M(x) + f_M(x)| \leq |f(x) - f_M(x)| + |f_M(x)| \Rightarrow$$
$$|f(x)| - |f_M(x)| \leq |f(x) - f_M(x)| < 1$$

Adding $|f_M(x)|$ to both sides of the outermost inequality gives us

$$|f(x)| < 1 + |f_M(x)|$$

This holds for any $x \in [a, b]$, so we have proven that f is bounded as well.

Let $\epsilon > 0$ be arbitrary. Again, since $f_n \to f$ uniformly, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{4(b-a)}$ (for all $n > N$, and all $x \in [a, b]$. Choose an $M > N$. Then f_M is integrable by assumption. By our integrability criterion, we know there is a partition $P_\epsilon \in \mathcal{P}[a, b]$ (let the number of components in the partition be $P := |P_\epsilon|$) such that

$$U(f_M, P_\epsilon) - L(f_M, P_\epsilon) = \sum_{k=1}^{p} M_{f_M,k}(x_k - x_{k-1}) - \sum_{k=1}^{p} M_{f_M,k}(x_k - x_{k-1}) = \sum_{k=1}^{p} (M_{f_M,K} - M_{f_M,K})(x_k - x_{k-1})$$

Now, since $M > N$, we have that $|f_M(x) - f(x)| < \frac{\epsilon}{4(a-b)}$ for all $x \in [a, b]$. For such x, by properties of absolute values,

$$f_M(x) - \frac{\epsilon}{4(b-a)} < f(x) < f_M(x) + \frac{\epsilon}{4(b-a)}$$

By the fact that every upper integral over a function is at least equal to the lower integral,

$$m_{f_M,k} - \frac{\epsilon}{4(b-a)} < m_{f,k} \leq M_{f,k} < M_{f_M,k} + \frac{\epsilon}{4(b-a)}.$$

By using the definitions of upper and lower integrals,

$$U(f, P_\epsilon) - L(f, P_\epsilon) = \sum_{k=1}^{p} M_{f,k}(x_k - x_{k-1}) - \sum_{k=1}^{p} m_{f,k}(x_k - x_{k-1})$$

Combine the sums: the above is equivalent to

$$\sum_{k=1}^{p} (M_{f,k} - m_{f,k})(x_k - x_{k-1})$$

By our above chain of inequalities,

$$\sum_{k=1}^{p}(M_{f,k}-m_{f,k})(x_k-x_{k-1}) \leq \sum_{k=1}^{p}\left(\left(M_{f_M,k}+\frac{\epsilon}{4(b-a)}\right)-\left(m_{f_M,k}-\frac{\epsilon}{4(b-a)}\right)(x_k-x_{k-1})\right) =$$

$$\sum_{k=1}^{p}(M_{f_M,k}-m_{f_M,k})(x_k-x_{k-1}+\sum_{k=1}^{p}\frac{2\epsilon}{4(b-a)}(x_k-x_{k-1}) <$$

$$\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

As a result, $U(f,P_\epsilon)-L(f,P_\epsilon)<\epsilon$ for an arbitrary ϵ. By our integrability criteria (Problem 2, Presentation Set 5), f is integrable on $[a,b]$. It remains to show that $\int_a^b f dx = \lim_{n\to\infty}\int_a^b f_n dx$. Let $\epsilon > 0$ be arbitrary. Since $f_n \to f$ uniformly, there exists an $N \in \mathbb{N}$ such that $|f_n(x)-f(x)| < \frac{\epsilon}{b-a}$ for all $n \in \mathbb{N}$ such that $n > N$ and $x \in [a,b]$. As a result of linearity of integration (Problem 6, Presentation Set 5):

$$\left|\int_a^b f_n dx - \int_a^b f dx\right| = \left|\int_a^b (f_n - f) dx\right|$$

for all such n. Then by Problem 7, Presentation Set 5, we obtain the absolute value bounds

$$\left|\int_a^b (f_n - f)dx\right| \leq \int_a^b |f_n - f| dx < \int_a^b \frac{\epsilon}{b-a}dx = (b-a)\frac{\epsilon}{b-a} = \epsilon$$

Since ϵ was arbitrary, we conclude that $\lim_{n\to\infty}\left|\int_a^b (f_n-f)dx\right| = 0$, and by properties of absolute values we acquire $\lim_{n\to\infty}\int_a^b f_n dx = \int_a^b f dx.$, as desired. □

9. Let $A \subset \mathbb{R}$ be countable. Then A has measure zero. (Note: the converse is not true.)

Proof. Suppose $A = \{x_n\}$ where $n = 1, 2, 3, ..., \infty$
Let $\epsilon > 0$, define open intervals

$I_n = (x_n - \frac{\epsilon}{2^{n+2}}, x_n + \frac{\epsilon}{2^{n+2}})$ where $n = 1, 2, 3, ..., \infty$

Then the length of each interval $\ell(I_n) = \frac{\epsilon}{2^{n+1}}$
And $A \subset \bigcup_{n=1}^{\infty} I_n$

We get $\sum_{n=1}^{\infty} \ell(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} < \epsilon$

Therefore if $A \subset \mathbb{R}$ is countable, then A has measure zero. $\square$

10. Let $\{A_k\}_{k=1}^{n}$ be a finite collection of sets of measure zero. Show that

$$\bigcup_{k=1}^{n} A_k$$

also has measure zero.

Proof. Let $\epsilon > 0$. Since each A_k is of measure zero, then for $\frac{\epsilon}{n}$, there exists a collection of open intervals $\{O_{k_m}\}$ for each A_k such that

$$A_k \subseteq \bigcup_{m=1}^{\infty} O_{k_m}$$

and

$$\sum_{m=1}^{\infty} |O_{k_m}| < \frac{\epsilon}{n}.$$

Then

$$\bigcup_{k=1}^{n} A_k \subseteq \bigcup_{k=1}^{n} \bigcup_{m=1}^{\infty} O_{k_m}$$

and

$$\sum_{k=1}^{n} \sum_{m=1}^{\infty} |O_{k_m}| < n(\frac{\epsilon}{n}) = \epsilon$$

because each collection of open intervals is countable, so the unions and summations are all well defined. Thus $\bigcup_{k=1}^{n} A_k$ has measure zero. $\square$

11. Let $f : [a, b] \mapsto \mathbb{R}$ be continuous. Then there exists $c \in (a, b)$ such that

$$\int_a^b f \, \mathrm{d}x = f(c)(b - a).$$

Proof. Let $f : [a, b] \mapsto \mathbb{R}$ be continuous.
Define

$$F(x) := \int_a^x f(t) \, \mathrm{d}t.$$

Thus, F is continuous on $[a, b]$.

If f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$. Thus for all $c \in (a, b)$, $F'(c) = f(c)$.

Since F is continuous on $[a, b]$ and differentiable on (a, b), we can invoke the Mean Value Theorem. Thus there exists $c \in (a, b)$ such that

$$F'(c) = \frac{F(b) - F(a)}{b - a}$$

$$= \frac{1}{b-a} \left(\int_a^b f(t)\ \mathrm{d}t - \int_a^a f(t)\ \mathrm{d}t \right) = \frac{1}{b-a} \int_a^b f(t)\ \mathrm{d}t.$$

So we have

$$f(c) = F'(c) = \frac{1}{b-a} \int_a^b f(t)\ \mathrm{d}t,$$

or

$$f(c)(b-a) = \int_a^b f(t)\ \mathrm{d}t.$$

□