Presentation Problems 5

21-355 A

Instructions: Your group should prepare a presentation for the problem corresponding to your group number. After presenting the solution and getting feedback from the class, you have until the beginning of the following class to send a $\TeX$ file with a polished form of the solution to the instructor. Make sure all group members' names are in the $\TeX$ file. There are 15 points available: 10 for the presentation and 5 for the written proof.

For these problems, assume all sets are subsets of $\mathbb{R}$ unless otherwise specified.

1. Let P and Q be partitions of $[a,b]$ such that $P \subseteq Q$. Then $U(f,P) \geq U(f,Q)$ and $L(f,P) \leq L(f,Q)$. Use this to show that for any partitions P_1 and P_2 of $[a,b]$ that $L(f,P_1) \leq U(f,P_2)$.

2. Let $f : [a,b] \mapsto \mathbb{R}$ be bounded. Then f is integrable on $[a,b]$ if and only if for all $\varepsilon > 0$, there exists some partition P_ε of $[a,b]$ such that

   $$U(f,P_\varepsilon) - L(f,P_\varepsilon) < \varepsilon.$$

3. Let $f : [a,b] \mapsto \mathbb{R}$ be bounded. Then f is integrable on $[a,b]$ if and only if there exists some sequence of partitions (P_n) of $[a,b]$ such that

   $$\lim_{n\to\infty} U(f,P_n) - L(f,P_n) = 0.$$

4. Let $f : [a,b] \mapsto \mathbb{R}$ be increasing. Then f is integrable on $[a,b]$.

5. Let $f : [a,b] \mapsto \mathbb{R}$ be continuous. Then f is integrable on $[a,b]$.

6. Let f, g be integrable on $[a,b]$ and let $\alpha, \beta \in \mathbb{R}$. Then $\alpha f + \beta g$ is integrable on $[a,b]$ and

   $$\int_a^b \alpha f + \beta g \ \mathrm{d}x = \alpha \int_a^b f \ \mathrm{d}x + \beta \int_a^b g \ \mathrm{d}x.$$

7. Let f, g be integrable on $[a,b]$

   (a) If $m \leq f(x) \leq M$ for all $x \in [a,b]$, then

   $$m(b-a) \leq \int_a^b f \ \mathrm{d}x \leq M(b-a).$$

(b) If $f(x) \leq g(x)$ on $[a, b]$, then

$$\int_a^b f \, \mathrm{d}x \leq \int_a^b g \, \mathrm{d}x.$$

(c) $|f|$ is integrable on $[a, b]$ and

$$\left| \int_a^b f \, \mathrm{d}x \right| \leq \int_a^b |f| \, \mathrm{d}x.$$

8. Let (f_n) be a sequence of real-valued functions on $[a, b]$ integrable on $[a, b]$. If $f_n \to f$ uniformly on $[a, b]$, then f is integrable on $[a, b]$ and

$$\lim_{n \to \infty} \int_a^b f_n \, \mathrm{d}x = \int_a^b f \, \mathrm{d}x.$$

9. Let $A \subset \mathbb{R}$ be countable. Then A has measure zero. (Note: the converse is not true.)

10. Let $\{A_k\}_{k=1}^n$ be a finite collection of sets of measure zero. Show that

$$\bigcup_{k=1}^n A_k$$

also has measure zero.

11. Let $f : [a, b] \mapsto \mathbb{R}$ be continuous. Then there exists $c \in (a, b)$ such that

$$\int_a^b f \, \mathrm{d}x = f(c)(b - a).$$