Presentation Problems 4

21-355 A

For these problems, assume all sets are subsets of $\mathbb{R}$ unless otherwise specified.

1. Let $f : A \mapsto \mathbb{R}$. If f is Lipschitz continuous, then f is uniformly continuous and if f is uniformly continuous, then f is continuous. Prove also that the reverse implications are not necessarily true.

   Proof. Since f is Lipschitz continuous, there exists $M > 0$ such that

   $$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M, \forall x \neq y \in A.$$

   Let $\epsilon \in \mathbb{R}^+$ and define $\delta = \frac{\epsilon}{M}$. Then, whenever $|x - y| < \delta$, we have

   $$\left|\frac{f(x) - f(y)}{x - y}\right| \leq M \implies |f(x) - f(y)| \leq M|x - y| < M\frac{\epsilon}{M} = \epsilon.$$

   Then f is uniformly continuous.

   Now we will prove f is continuous. Take $c \in A$ and $\epsilon \in \mathbb{R}^+$. Define $\delta = \frac{\epsilon}{M}$. Then, whenever $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$ as presented above. Therefore, f is continuous.

   Now, we will prove by a counterexample that a continuous function is not necessarily uniformly continuous. Consider the function f defined on $A = (0, +\infty)$ by

   $$f(x) = \frac{1}{x}$$

   Let $\epsilon > 0$ and $x_0 \in A$. Then select $\delta = \min\{x_0/2, (x_0/2)^2\epsilon\}$. Then we know that $\delta \leq x_0/2$, and $\delta \leq (x_0/2)^2\epsilon$.

   Now let $x \in A$. Then whenever $|x - x_0| < \delta$, we have that

   $$x_0 - x \leq |x - x_0| < \delta \leq \frac{x_0}{2} \implies x_0 - x < \frac{x_0}{2} \implies x > \frac{x_0}{2}$$

   Since $x_0 > x_0/2$ and $x_0 > 0$, we have that $x \cdot x_0 > (x_0/2)^2$. Thus

   $$|f(x) - f(x_0)| = |\frac{1}{x} - \frac{1}{x_0}| = \frac{|x - x_0|}{x \cdot x_0} < \frac{\delta}{(x_0/2)^2} \leq \frac{(x_0/2)^2\epsilon}{(x_0/2)^2} = \epsilon$$

Therefore, $f(x)$ is continuous on A.

Now we want to show there there exists $\epsilon > 0$ such that for $\forall\delta > 0, \exists x, y \in A$, for $|x - y| < \delta, |f(x) - f(y)| \geq \delta$. Let $\epsilon = 1$ and $\delta > 0$. Pick $x = \min\{\delta, 1\}$ and $y = x/2$. Therefore, $|x - y| = x/2 \leq \delta$. Then we have

$$|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{x/2}| = \frac{1}{x} \geq 1 = \epsilon$$

Thus, we have shown that $f(x) = 1/x$ is continuous but not uniformly continuous on $(0, +\infty)$ ☐

2. Let K be compact and let $f : K \mapsto \mathbb{R}$ be continuous on K. Then f is uniformly continuous on K.

   Proof. We show the contrapositive. If f is not uniformly continuous on K, by definition there exists $\epsilon > 0$ such that for all $\delta > 0$ there exist $x, y \in K$ such that $|x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$. Thus, for each $n \in \mathbb{N}$, choose $x_n, y_n \in K$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon$. For all $\epsilon' > 0$ and $n > \frac{1}{\epsilon'}$, $|x_n - y_n| < \frac{1}{n} < \epsilon'$, so $|x_n - y_n| \to 0$.

   Since K is compact, the sequence (x_n) in K has a subsequence $(x_{n_k}) \to x \in K$. If (y_{n_k}) is the subsequence whose indices correspond to those in (x_{n_k}), it has a subsequence $(y_{n_{k_j}}) \to y \in K$. The corresponding sequence $(x_{n_{k_j}})$ converges to $\lim(x_{n_k}) = x$. Thus $\lim(y_{n_{k_j}}) = \lim((y_{n_{k_j}} - x_{n_{k_j}}) + x_{n_{k_j}}) = \lim(y_{n_{k_j}} - x_{n_{k_j}}) + \lim(x_{n_{k_j}}) = \lim(x_{n_{k_j}}) = x$.

   By Presentation 3 Problem 8, if f is continuous, $\lim f(x_{n_{k_j}}) = \lim f(y_{n_{k_j}}) = f(x)$ and $\lim(f(x_{n_{k_j}}) - f(y_{n_{k_j}}) = 0$. Therefore, there exists an $n' = n_{k_j}$ such that $|f(x_{n'}) - f(y_{n'})| < \epsilon$, which contradicts our choice of x_ns and y_ns at the beginning. Thus f is not continuous.

   Therefore, if f is not uniformly continuous, f is not continuous and the original statement follows. ☐

3. Let K be compact and $f : K \mapsto \mathbb{R}$ be continuous on K. Then $f(K)$ is compact in $\mathbb{R}$.

   Proof. Since Take a sequence $(y_n) \subseteq f(K)$ pick some sequence in K, denoted x_n, $x_n \in K$ such that $f(x_n) = y_n$ for each $n \in \mathbb{N}$.
   Since K is compact, we could find some subsequence (x_{n_i}) of (x_n) converging to some x in K.
   $\lim_{i\to\infty} x_{n_i} = x$, where $x \in K$
   Since f is continuous, $\lim_{i\to\infty} f(x_{n_i}) = f(x)$, $f(x) \in f(K)$
   $\lim_{i\to\infty} y_{n_i} = y$, $y \in f(K)$

   ☐

4. Let $f : E \mapsto \mathbb{R}$ be continuous on E and E be connected. Then $f(E)$ is connected.

Proof. We use the theorem that the only connected sets in $\mathbb{R}$ are intervals.
WTS: $\forall\ a, b \in f(E), c \in \mathbb{R}$ s.t. $a < c < b$, $c \in f(E)$
Let $A = \{e \in E : f(e) < c\}$, $B = \{e \in E : f(e) \geq c\}$
Then A and B are disjoint, non-empty, and $E = A \cup B$
Since E is connected, $\exists$ sequence $(x_n) \in A$ s.t. $x_n \mapsto x \in B$
Since f is continuous on E, $f(x_n) \mapsto f(x)$
Then $\forall n \in \mathbb{N}$, $f(x_n) < c$. So $f(x) \leq c$.
But $f(x) \geq c$ since $x \in B$. So $f(x) = c \implies c \in f(E)$
Therefore, for any $a, b \in f(E), c \in \mathbb{R}$ s.t. $a < c < b$, we have $c \in f(E)$.
Then $f(E) \subseteq \mathbb{R}$ is an interval, thus is connected. □

5. Let (f_n) be a sequence of functions mapping A to $\mathbb{R}$. If (f_n) is uniformly Cauchy, then (f_n) converges uniformly.

Proof. WTS: Uniformly Cauchy $\implies$ Uniformly Convergent.
Because the sequences of function is uniformly Cauchy,we know that $\forall x \in R, (f_n(x))$ is a Cauchy sequence, which subsequently implies converges.
Therefore, $\forall x \in R, f_n(x) \to L_x$ for some $L_x \in \mathbb{R}$.
Now we construct $f(x)$ using L_x.
From Uniformly Cauchy, we know
$\forall\ \epsilon > 0, \exists\ N$ s.t. $\forall\ m, n \geq N$, $|f_m(x) - f_n(x)| < \epsilon\ \forall\ x \in \mathbb{R}$.
$\implies -\epsilon < f_m(x) - f_n(x) < \epsilon\ \forall x$
Fixing an x
Taking the limit as $n \to \infty$.
We also know that $\lim_{n\to\infty} f_n(x) = f(x)$
$\implies -\epsilon < f_m(x) - f(x) < \epsilon$.
$\implies |f_m(x) - f(x)| < \epsilon\ \forall\ x$, which is the definition of Uniformly Convergent.

□

6. Let (f_n) be a sequence of real-valued functions continuous on A. If (f_n) converges uniformly, then (f_n) converges pointwise to the same uniform limit function f and f is continuous on A.

Proof. Let (f_n) be a sequence of real-valued functions continuous on A, and (f_n) converges uniformly to f.

$\Rightarrow$ Since it converges uniformly, we know $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. if $n \geq N, |f_n(x) - f(x)| < \epsilon$ holds to be true for all $x \in A$.

$\Rightarrow \forall \epsilon > 0$, for each $x \in A$, if $n \geq N, |f_n(x) - f(x)| < \epsilon$, which means $(f_n(x))$ is pointwise convergent.

Now we'd like to show $f(x)$ is continuous on A.

Let $c \in A$.

In addition, because we know (f_n) converges uniformly, we know $\forall \epsilon > 0, \forall x \in A, \exists N \in \mathbb{N}$ s.t. if $n \geq N, |f_n(x) - f(x)| < \frac{\epsilon}{3}$. Let N be fixed.

Moreover, in previous part, we have shown (f_n) is also pointwise convergent, thus $\forall \epsilon > 0$, with the same N we found in above, we know $\forall n \geq N, |f_n(c) - f(c)| < \frac{\epsilon}{3}$.

By assumption, we know $f_n(x)$ is continuous everywhere on A, which means $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $|x - c| < \delta$, $|f_n(x) - f_n(c)| < \frac{\epsilon}{3}$.

Put everything together: $\forall \epsilon > 0, \exists \delta > 0$ s.t. if $|x - c| < \delta$ when $n \geq N$:

$$\begin{aligned}
|f(x) - f(c)| &= |f(x) - f_n(x) + f_n(x) - f_n(c) + f_n(c) - f(c)| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\
&\quad \text{by triangle inequality} \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon
\end{aligned}$$

We can conclude, f is continuous on A. □

7. Let $f_n : [0, 1] \mapsto \mathbb{R}$ where $f_n(x) = x^n$ for each $n \in \mathbb{N}$. Show that (f_n) does not converge uniformly, but does converge pointwise.

   Proof. If $0 \leq x < 1$, then $x^n \to 0$ and $n \to \infty$, since $x < 1$, then (x_n) is a polynomial in the form of x^n, and clearly every sequence will converge to 0. If $x = 1$, then $x^n \to 1$ as $n \to \infty$ since every power of 1 is 1. So $f_n \to f$ pointwise where

$$f(x) = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ \\ 1, & \text{if } x = 1 \end{cases}$$

   AFSOC that (f_n) converges uniformly. Then using the results proved in Presentation 4 Problem 6, its limit must be the same as the pointwise limit. However, the pointwise limit is not continuous, so (f_n) cannot converge uniformly. □

8. Let f be defined on (a, b) for some $a < b$ and let f be differentiable at $c \in (a, b)$. Then f is continuous at c

Proof. By the definition of differentiability at a point, the derivative of f at c exists and we call it $f'(c)$. In particular,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c)$$

We can multiply both sides of the equation by $\lim_{x \to c}(x - c)$:

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \cdot \lim_{x \to c}(x - c) = f'(c) \lim_{x \to c}(x - c)$$

Both limits on the left-hand side exist, so we can rewrite this equation as

$$\lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \cdot (x - c) \right) = f'(c) \lim_{x \to c}(x - c) \Rightarrow$$
$$\lim_{x \to c}(f(x) - f(c)) = f'(c) \lim_{x \to c}(x - c)$$

By the algebraic limit theorem for addition (or subtraction), we can split the subtraction expressions on both sides of the equation to obtain

$$\lim_{x \to c} f(x) - \lim_{x \to c} f(c) = f'(c) \left(\lim_{x \to c} x - \lim_{x \to c} c \right)$$

Adding $\lim_{x \to c} f(c)$ to both sides of the equation gives

$$\lim_{x \to c} f(x) = \lim_{x \to c} f(c) + f'(c) \left(\lim_{x \to c} x - \lim_{x \to c} c \right).$$

Then by direct substitution,

$$\lim_{x \to c} f(x) = f(c) + f'(c)(c - c) \Rightarrow$$
$$\lim_{x \to c} f(x) = f(c)$$

The result of Problem 7 from Presentation Set 3 tells us that because the above is true, f is continuous at c, as desired. □

9. Let $f : (a, b) \mapsto \mathbb{R}$ for some $a < b$. f is differentiable at $c \in (a, b)$ if and only if there exists some function L on (a, b) continuous at c such that for all $x \in (a, b)$,

   $f(x) - f(c) = L(x)(x - c)$.

Proof. Forward direction:
Assume that $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists.
Define $L(x) = \frac{f(x)-f(c)}{x-c}$ for $x \neq c$, and $L(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ for x = c.
Let $x_0 \in (a,b)$, and there are two cases.
Case 1: $x_0 \neq c$
Then $L(x_0) = \frac{f(x_0)-f(c)}{x_0-c}$ and $\lim_{x\to x_0} f(x) - f(c) = f(x_0) - f(c)$
Then $\lim_{x\to x_0} = x_0 - c$ and $x_0 - c \neq 0$ because $x_0 \neq c$
Therefore, $\lim_{x\to x_0} L(x) = \lim_{x\to x_0} \frac{f(x)-f(c)}{x-c} = \frac{f(x_0)-f(c)}{x_0-c} = L(x_0)$
Case 2: $x_0 = c$
$\lim_{x\to c} L(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L(c)$ by definition of L.

Backward direction:
For $x \neq c$, $L(x) = \frac{f(x)-f(c)}{x-c}$.
So $\lim_{x\to c} L(x) = L(c)$ because L is continuous
Which means $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. □

10. Let f and g be defined on (a,b) for $a < b$ and differentiable at $c \in (a,b)$. Show that

(a) $f+g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$

Proof. By definition of differentiability, $(f+g)'(c) = \lim_{x\to c} \frac{(f+g)(x)-(f+g)(c)}{x-c}$ provided the limit exists. We will show the limit exists and is equal to $f'(c) + g'(c)$.

$$
\begin{aligned}
(f+g)'(c) &= \lim_{x\to c} \frac{(f+g)(x)-(f+g)(c)}{x-c} && \text{Def of differentiability}\\
&= \lim_{x\to c} \frac{f(x)+g(x)-f(c)-g(c)}{x-c} && \text{Def of } (f+g)(x)\\
&= \lim_{x\to c} \frac{f(x)-f(c)+g(x)-g(c)}{x-c} &&\\
&= \lim_{x\to c} \frac{f(x)-f(c)}{x-c} + \lim_{x\to c} \frac{g(x)-g(c)}{x-c} && \text{Algebraic Limit Theorem}\\
&= f'(c) + g'(c) && f,\ g \text{ differentiable at } c
\end{aligned}
$$

□

(b) for any $k \in \mathbb{R}$, kf is differentable at c and $(kf)'(c) = kf'(c)$.

Proof. By definition of differentiability, $(kf)'(c) = \lim_{x\to c} \frac{(kf)(x)-(kf)(c)}{x-c}$ provided the limit exists. We will show the limit exists and is equal to $kf'(c)$.

$$\begin{aligned}(kf)'(c) &= \lim_{x\to c}\frac{(kf)(x)-(kf)(c)}{x-c} && \text{Def of differentiability}\\ &= \lim_{x\to c}\frac{kf(x)-kf(c)}{x-c} && \text{Def of } (kf)(x)\\ &= \lim_{x\to c} k\frac{f(x)-f(c)}{x-c} \\ &= k\lim_{x\to c}\frac{f(x)-f(c)}{x-c} && \text{Algebraic Limit Theorem}\\ &= kf'(c) && f \text{ differentiable at } c\end{aligned}$$

□

11. Let f and g be defined on (a,b) for $a<b$ and differentiable at $c\in(a,b)$. Show that

    (a) (fg) is differentiable at c and $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$

    (b) if $g(c)\neq 0$, then $\left(\frac{f}{g}\right)$ is differentiable at c and

$$\left(\frac{f}{g}\right)(c)=\frac{f'(c)g(c)-f(c)g'(c)}{[g(c)]^2}.$$

Proof. Let f and g be defined on (a,b) for $a<b$ and differentiable at $c\in(a,b)$.
Show that

(a) (fg) is differentiable at c and $(fg)'(c)=f'(c)g(c)+f(c)g'(c)$.
$(fg)'(c)=\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c}=\lim_{x\to c}\frac{f(x)(g(x)-g(c))+g(c)(f(x)-f(c)}{x-c}$
$=\lim_{x\to c}\frac{f(x)(g(x)-g(c))}{x-c}+\lim_{x\to c}\frac{g(c)(f(x)-f(c))}{x-c}$
$=\lim_{x\to c}f(x)\lim_{x\to c}\frac{g(x)-g(c)}{x-c}+g(c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}$
$=f(c)g'(c)+g(c)f'(c)$

(b) if $g(c)\neq 0$, then $(\frac{f}{g})$ is differentiable at c and

$$(\frac{f}{g})'(c)=\frac{f'(c)g(c)-f(c)g'(c)}{[g(c)]^2}$$

$(\frac{f}{g})'(c)=\lim_{x\to c}\frac{\frac{f(x)}{g(x)}-\frac{f(c)}{g(c)}}{x-c}=\lim_{x\to c}\frac{f(x)g(c)-g(x)f(c)}{g(x)g(c)(x-c)}$
$=\lim_{x\to c}\frac{1}{g(x)g(c)}\lim_{x\to c}\frac{f(x)g(c)-g(x)f(c)}{x-c}$
$=\frac{1}{[g(c)]^2}[\lim_{x\to c}g(c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}-\lim_{x\to c}f(x)\lim_{x\to c}\frac{g(x)-g(c)}{x-c}]$
$=\frac{1}{g(c)^2}[g(c)f'(c)-f(c)g'(c)]$

Note: $\lim_{x\to c}f(x)=f(c)$ and $\lim_{x\to c}g(x)=g(c)$. Since f,g are both differentiable at c, they must both be continuous at c. □