Presentation Problems 4

21-355 A

Instructions: Your group should prepare a presentation for the problem corresponding to your group number. After presenting the solution and getting feedback from the class, you have until the beginning of the following class to send a TEX file with a polished form of the solution to the instructor. Make sure all group members' names are in the TEX file. There are 15 points available: 10 for the presentation and 5 for the written proof.

For these problems, assume all sets are subsets of $\mathbb{R}$ unless otherwise specified.

1. Let $f : A \mapsto \mathbb{R}$. If f is Lipschitz continuous, then f is uniformly continuous and if f is uniformly continuous, then f is continuous. Prove also that the reverse implications are not necessarily true.

2. Let K be compact and let $f : K \mapsto \mathbb{R}$ be continuous on K. Then f is uniformly continuous on K.

3. Let K be compact and $f : K \mapsto \mathbb{R}$ be continuous on K. Then $f(K)$ is compact in $\mathbb{R}$.

4. Let $f : E \mapsto \mathbb{R}$ be continuous on E and E be connected. Then $f(E)$ is connected.

5. Let (f_n) be a sequence of functions mapping A to $\mathbb{R}$. If (f_n) is uniformly Cauchy, then (f_n) converges uniformly.

6. Let (f_n) be a sequence of real-valued functions continuous on A. If (f_n) converges uniformly, then (f_n) converges pointwise to the same uniform limit function f and f is continuous on A.

7. Let $f_n : [0, 1] \mapsto \mathbb{R}$ where $f_n(x) = x^n$ for each $n \in \mathbb{N}$. Show that (f_n) does not converge uniformly, but does converge pointwise.

8. Let f be defined on (a, b) for some $a < b$ and let f be differentiable at $c \in (a, b)$. Then f is continuous at c

9. Let $f : (a, b) \mapsto \mathbb{R}$ for some $a < b$. f is differentiable at $c \in (a, b)$ if and only if there exists some continuous function L on (a, b) such that for all $x \in (a, b)$,

   $$f(x) - f(a) = L(x)(x - a).$$

10. Let f and g be defined on (a, b) for $a < b$ and differentiable at $c \in (a, b)$. Show that

    (a) $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$

    (b) for any $k \in \mathbb{R}$, kf is differentable at c and $(kf)'(c) = kf'(c)$.

11. Let f and g be defined on (a, b) for $a < b$ and differentiable at $c \in (a, b)$. Show that

    (a) (fg) is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

    (b) if $g(c) \neq 0$, then $\left(\frac{f}{g}\right)$ is differentiable at c and

$$\left(\frac{f}{g}\right)(c) = \frac{f'(c)g(c) - f(c)g'(c)}{[g(c)]^2}.$$