Presentation Problems 3

21-355 A

1. Show that $K \subseteq \mathbb{R}$ is compact if and only if K is closed and bounded in $\mathbb{R}$.

   Proof. ($\Longrightarrow$) Let K be compact.

   (Bounded)

   AFSOC that K is not bounded. Then we know that there exists $x_1 \in K$ satisfying $|x_1| > 1$. Similarly, there exists $x_2 \in K$ with $|x_2| > 2$. In general, $\forall n \in \mathbb{N}$, there exists $x_n \in K$, such that $|x_n| > n$. Define (x_n) to be the sequence obtained via the above procedure. We know (x_n) is a sequence in K. Since K is assumed to be compact, (x_n) should have a convergent subsequence (x_{n_k}). But the elements of the subsequence must satisfy $|x_{n_k}| > n_k$, and consequently (x_{n_k}) is unbounded. Since we have shown that every convergent sequence is bounded, the sequence (x_{n_k}) is not convergent, a contradiction.

   (Closed)

   We will show that K is closed by proving that K contains its limit points.

   Let $x = \lim x_n$, where (x_n) is a convergent sequence in K. By the definition of a compact set, the sequence (x_n) has a convergent subsequence (x_{n_k}), and this subsequence converges to the same limit x by problem 1 of presentation set 1. Therefore, it follows that $x \in K$, and hence K is closed.

   ($\Longleftarrow$) Let K be closed and bounded in $\mathbb{R}$. Let (x_n) be a sequence in K. Since this sequence is bounded, we have a converging subsequence (x_{n_k}) in $K \subseteq \mathbb{R}$. Let $x = \lim x_{n_k}$, we have $x \in K$ since K is closed. Therefore, we know that $K \subseteq \mathbb{R}$ is compact. $\square$

2. Let $\{K_n\}_{n=1}^{\infty}$ be a collection of non-empty compact sets in $\mathbb{R}$ such that for each $n \in \mathbb{N}$, $K_n \supseteq K_{n+1}$. Then

$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

Proof. For each $n \in \mathbb{N}$, choose $x_n \in K_n$. Since $K_n \subseteq K_1$ for each $n \in \mathbb{N}$, (x_n) is a sequence in K_1; since K_1 is compact, there is a subsequence (x_{n_k}) converging to some $x \in K_1$.

Moreover, for each $n \in \mathbb{N}$, the subsequence $\{x_{n_k} : n_k \geq n\}$ also converges to x. By the first presentation problem, the compact set K_n is also closed; consequently, by the Presentation 2 Problem 2, $x \in K_n$.

Therefore, $x \in K_n$ for all $n \in \mathbb{N}$, so $\bigcap_{n=1}^{\infty} K_n = \emptyset$. □

3. Show that any non-empty perfect set in $\mathbb{R}$ is uncountable. Conclude that any non-empty perfect set must contain irrationals.

Proof. Assume for the sake of contradiction, some non-empty perfect set P is countable.
so we can write $P = \{p_1, p_2, p_3, ...\}$.

Let $I_1 = (p_1 - \epsilon, p_1 + \epsilon)$.
Since p_1 is limit point, there must be another element of P in I_1. Without loss of generality, call it p_2.

We construct $I_2 = (p_2 - a, p_2 + a)$ such that $\overline{I_2} \subseteq I_1$
Again, because p_2 is a limit point, $p_3 \in I_2$.
Construct $\overline{I_3} \subseteq I_2$, but $p_1, p_2 \notin I_3$.

Thus, for each n, $\overline{I_{n+1}} \subseteq I_n$, $p_1, ...p_n \notin I_{n+1}$, but $p_{n+1} \in I_{n+1}$.
Set $P' = \bigcap_{p_i \in P}(\overline{I_i} \cap P)$, where $\overline{I_i}$ is closed and bounded, so it is compact
P' is non-empty, since $\overline{I_(i+1)} \cap P \subseteq \overline{I_i} \cap P$ yet $p_1, p_2... \notin \overline{I_i} \cap P$

This contradicts our assumption that P is countable. So P is uncountable. If P only contained rationals, it would be countable. Since it is uncountable, P must contain irrationals.

□

4. Show that a set $A \subseteq \mathbb{R}$ is connected if and only if $a < c < b$ with $a, b \in A$, then $c \in A$. Use this to show the only connected sets in $\mathbb{R}$ are intervals.

Proof. $(\Rightarrow)$: Assume that set $A \in \mathbb{R}$ is connected. We want to show that if $a < c < b$ with $a, b \in A$, then $c \in A$. Equivalently, we will show the contrapositive: if $a < c < b$ with $a, b \in A$ and $c \notin A$, then A is not connected.
Suppose that $a < c < b$ with $a, b \in A$ and $c \notin A$. Let $A_1 = A \cap (-\infty, c)$, $A_2 = A \cap (c, \infty)$. Then A_1, A_2 are non-empty and disjoint, $A_1 \cup A_2 = A$. Then

$$\overline{A_1} \cap A_2 \subseteq \overline{(-\infty, c)} \cap (c, \infty) = (-\infty, c] \cap (c, \infty) = \emptyset$$
$$A_1 \cap \overline{A_2} \subseteq (-\infty, c) \cap \overline{(c, \infty)} = (-\infty, c) \cap [c, \infty) = \emptyset$$

So $A_1, A_2 \subseteq A$ are separated, and therefore, A is not connected.
$(\Leftarrow)$: Assume that if $a < c < b$ with $a, b \in A$, then $c \in A$, we want to show A is connected.
Write $A = A_1 \cup A_2$ where A_1 and A_2 are non-empty and disjoint. Let $a_1 \in A_1$ and $b_1 \in A_2$.
WLOG, $a_1 < b_1$. By assumption, for any c satisfies $a_1 < c < b_1$, $c \in A$. Then $[a_1, b_1] \subseteq A$.
Let $I_1 = [a_1, b_1]$. Let $c_1 = \frac{a_1+b_1}{2}$. If $c_1 \in A_1$, let $a_2 = c_1, b_2 = b_1$. Otherwise, let $a_2 = a_1, b_2 = c_1$. Then let $I_2 = [a_2, b_2]$. We continue the construction inductively. It is guaranteed that all a_ns are in A_1, and all b_ns are in A_2, and $[a_1, b_1] \supseteq [a_2, b_2] \supseteq .... \supseteq [a_n, b_n]$..... Then by the nested interval property $\exists x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$, and x is either in A_1 or A_2. Besides, $(a_n) \to x$ and $(b_n) \to x$.

- if $x \in A_1$, since $(b_n) \to x$ and $b_n \in A_2$ for all $n \in \mathbb{N}$, A is connected.
- if $x \in A_2$, since $(a_n) \to x$ and $a_n \in A_1$ for all $n \in \mathbb{N}$, A is connected.

Therefore, if $a < c < b$ with $a, b \in A \Rightarrow c \in A$, A is connected.

Since we have shown that if A is connected, then $a < c < b$ with $a, b \in A \Rightarrow c \in A$ and if $a < c < b$ with $a, b \in A$ and $c \notin A$, then A is not connected, the only connected sets in $\mathbb{R}$ are intervals.

□

5. Let $f : A \mapsto \mathbb{R}$ and let c be a limit point of A. Then $\lim_{x \to c} f(x) = L$ if and only if for all sequences (x_n) in A such that $x_n \neq c$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n = c$, $\lim_{n\to\infty} f(x_n) = L$.

Proof. $(\Rightarrow)$Let $f : A \to \mathbb{R}$ and let c be a limit point of A, and $\lim_{x\to c} f(x) = L$
We know $\forall \epsilon^* > 0, \exists\ \delta > 0$ s.t. If $0 < |x - c| < \delta$,then $|f(x) - L| < \epsilon^*$. Thus, we know for any $\epsilon^* > 0$, such a δ exists.
Let (x_n) be a sequence in A such that $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$. Now, let $\epsilon = \delta$. We also know that $\forall\ \delta > 0$, $\exists\ N_1$ s.t. $\forall\ n \geq N_1$, $|x_n - c| < \delta$. Thus, combining the two givens, we know that for any x_n s.t. $|x_n - c| < \delta$, which is $x_n | n \geq N_1$, we know that $|f(x_n) - L| < \epsilon^*$.

So, we know that given the two conditions, $\forall \epsilon > 0$, $\exists\ N$ s.t. $\forall\ n \geq N$, $|f(x_n) - L| < \epsilon$.
($\Leftarrow$)For the 2nd direction, we are going to prove the contrapositive. Assume $\lim_{x\to c} f(x) \neq L$. Then that means either the limit d.n.e. or the limit $\neq L$. So, there exists some $\epsilon > 0$, such that for all $\delta > 0$ such that $\exists\ x^*$ s.t. $0 < |x^* - c| < \delta$, but $|f(x^*) - L| \geq \epsilon$.
The intuition behind this proof is that no matter how close an x gets to c, f(x) could still be infinitely far from L.
Define x_n to be a point in A not equal to c such that $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon$. Clearly, $x_n \to c$ but $(f(x_n))$ does not converge to L.

□

6. Let $f, g : A \mapsto \mathbb{R}$ such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then

   (a) $\lim_{x\to c}[\alpha f(x) + \beta g(x)] = \alpha L + \beta M$ for all $\alpha, \beta \in \mathbb{R}$,

   (b) $\lim_{x\to c}[f(x)g(x)] = LM$, and

   (c) if $M \neq 0$,
   $$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

   Proof. Recall the theorem from class: Let $f : A \to \mathbb{R}$ and x_0 be a limit point of A. Then $\lim\limits_{x\to x_0} f(x) = L$ if and only if $f(x_n) \to L$ for all $(x_n) \subseteq A$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$. We can use this with f and g since f with c and g with c both satisify the conditions.

   Since $\lim\limits_{x\to c} f(x) = L$, then by the above theorem, we know that $f(x_n) \to L$ for all $(x_n) \subseteq A$ such that $x_n \neq c$ for all $n \in \mathbb{N}$ and $x_n \to c$. The same goes for g, so $g(y_n) \to M$ for all $(y_n) \subseteq A$ such that $y_n \neq c$ for all $n \in \mathbb{N}$ and $y_n \to c$. Let (x_n) be a fixed sequence that satisfies this property for f and g (we know this exists exist because c is a limit point in A).

   By presentation problem 2 in the first set of presentation problems, we know if (a_n) and (b_n) are sequences such that $\lim a_n = a$ and $\lim b_n = b$, then for any $\alpha, \beta \in \mathbb{R}$, $\lim(\alpha a_n + \beta b_n) = \alpha a + \beta b$, $\lim(a_n b_n) = ab$, and $\lim \frac{a_n}{b_n} = \frac{a}{b}$, given $b \neq 0$
   Thus we can conclude:

   (a) $\lim\limits_{x\to c}[\alpha f(x) + \beta g(x)] = \alpha L + \beta M$,

   (b) $\lim\limits_{x\to c}[f(x)g(x)] = LM$,

   (c) and provided $M \neq 0$, $\lim\limits_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$.

□

7. Let $A \subseteq \mathbb{R}$ and let $c \in A$. Then $f : A \mapsto \mathbb{R}$ is continuous at c if and only if for all sequences (x_n) in A such that $x_n \to c$, $f(x_n) \to f(c)$.

   Proof. Let $A \subseteq \mathbb{R}$ and let $c \in A$. Let $f : A \to \mathbb{R}$.

   For the forwards direction, suppose f is continuous at c. Let (x_n) be a sequence in A such that $x_n \to c$. We will show that $f(x_n) \to f(c)$. Let $\varepsilon > 0$. Since f is continuous at c, we know there exists $\delta > 0$ such that if $|x_n - c| < \delta$ then $|f(x_n) - f(c)| < \varepsilon$. Since $\delta > 0$ and $x_n \to c$, we know there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have

   $$|x_n - c| < \delta$$

   It follows that from continuity that for all $n \geq N$,

   $$|f(x_n) - f(c)| < \varepsilon$$

   Thus $f(x_n) \to f(c)$.

   Will we show the other direction by contrapositive. Suppose f is not continuous at c. This means that there exists some $\varepsilon > 0$ such that for all $\delta > 0$, if $|x - c| < \delta$ then $|f(x) - f(c)| \geq \varepsilon$. Then we can construct a sequence (x_n) in A such that $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \varepsilon$. We can see that $x_n \to c$. However, for any $\varepsilon > 0$, we have that $|f(x_n) - f(c)| \geq \varepsilon$, so $f(x_n) \not\to f(c)$.

   □

8. Let $A \subseteq \mathbb{R}$ and $f : A \mapsto \mathbb{R}$. Then f is continuous on A if and only if for all open sets U in $\mathbb{R}$, $f^{-1}(U)$ is open in A. (Recall that a set O is open in A if and only if for all $x \in O$, there is some $r > 0$ such that

   $$B_A(x, r) := \{y \in A : |x - y| < r\}$$

   is contained in O.)

   Proof. We will first prove the forward direction; that is, we assume f is continuous at A and want to show that for all open sets U in $\mathbb{R}$, $f^{-1}(U)$ is open in A. Let U be an arbitrary open set in $\mathbb{R}$. Since f is continuous on A, it is continuous at any $x \in A$. By the definition of continuity, for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$.

   Let $y \in f^{-1}(U)$ be arbitrary. Then obviously $f(y) \in U$. Since U is open, there exists an $r > 0$ such that $B(f(y), r) \subseteq U$. Set $\epsilon = r$. Choose δ such that $|y - x_0| < \delta \Rightarrow |f(y) - f(x_0)| < \epsilon$ (which exists due to f's continuity). As a result, for all $x \in B(y, \epsilon)$, we see $f(x) \in B(f(y), \epsilon) = B(f(y), r) \subseteq U$. By the definition of preimage, $x \in f^{-1}(U)$. Then $B(y, \delta) \subseteq f^{-1}(U)$, so $f^{-1}(U)$ is open in A. Since U was arbitrary, we are done with this direction.

Now we prove the reverse direction; that is, we assume for all open sets $U \subseteq \mathbb{R}$ that $f^{-1}(U)$ is open in A. We want to show that f is continuous on A. It suffices to show f is continuous at x for all $x \in A$, and we let this x be arbitrary to prove it for all such x at once. We want to show for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in A$, if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. Let $\epsilon > 0$ be arbitrary. Then $B(f(x), \epsilon)$ is open in $\mathbb{R}$ (it was proven in lecture that open balls are open). An equivalent statement is that $\{x_0 \in A : f(x_0) \in B(f(x), \epsilon)\}$ is open in A (by the definition of a preimage). By the definition of an open ball, this set is equal to $S := \{x_0 \in A : |f(x) - f(x_0)| < \epsilon\}$, which is open in A.

This means, by the definition of open, that for all $y \in S$, there exists a $\delta > 0$ such that $B_A(y, \delta) \subseteq S$. Then there exists a $\delta > 0$ such that $\{a \in A : |y - a| < \delta\} \subseteq S$. This means that if $|y - a| < \delta$, then $|f(y) - f(a)| < \epsilon$. Since ϵ was arbitrary, f is continuous at $x \in A$. Since $x \in A$, was arbitrary, f is continuous on A, as desired.

□

9. Let $A \subseteq \mathbb{R}$ and $f : A \mapsto \mathbb{R}$. Then f is continuous at every isolated point of A.

   Proof. Take an arbitrary isolated point x in A. By definition of isolatated point, there exist a $r > 0$ such that $B(x, r) \cap x = x$.
   For all $\epsilon > 0$, pick positive $\delta = r/2$.
   For any y in A such that $|x - y| < \delta$, y has to equal to x since it's in $B(x, \delta)$, and $B(x, \delta) \subset B(x, r)$.
   That implies, $|f(x) - f(y)| = |f(x) - f(x)| = 0$, which is less than any $\epsilon > 0$. This proves that f is continuous at x, and since x is arbitrary, this applies to any isolated points in A. □

10. Let A, B be subsets of $\mathbb{R}$ such that $f : A \mapsto \mathbb{R}$, $g : B \mapsto \mathbb{R}$ and $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

    Proof. Let $c \in A$. Since f is continuous at c, by the sequential definition of continuity, for any arbitrary sequence $(x_n) \subseteq A$ such that $x_n \to c$, we have $f(x_n) \to f(c)$ since $f(x_n) \subseteq f(A) \subseteq B$. Also, since g is continuous at $f(c)$, by the sequential definition of continuity we have that if $f(x_n) \to f(c)$, then $g(f(x_n)) \to g(f(c))$. Thus, since $g(f(c)) = (g \circ f)(c)$, and we have for any $x_n \to c$ that $(g \circ f)(x_n) \to (g \circ f)(c)$, then $g \circ f$ is continuous at c. □

11. Let $p(x)$ the polynomial

$$p(x) = \sum_{k=0}^{n} a_k x^k.$$

Then p is continuous on $\mathbb{R}$.

Proof. To prove this we will build up $p(x)$ from continuous parts. An integral part to completing this proof will require us to recall that a function f is continuous at c if

$$\lim_{x\to c} f(x) = f(c)$$

. First fix c an arbitrary element of $\mathbb{R}$.

First we will prove that if $f(x) = x$ then $f(x)$ is continuous at c. Fix $\epsilon > 0$. Let $\delta = \epsilon$. If $|x - c| < \delta$ Then it follows that:

$$|f(x) - f(c)| = |x - c| < \delta = \epsilon$$

So $|f(x)-f(c)| < \epsilon$, therefore $f(x) = x$ is continuous at c, and $\lim_{x\to c} f(x) = f(c)$.

From presentation problem 6, we have the limit product rule. So we can use induction to show that $f(x) = x^k$ is continuous at c. We already showed the base case $f(x) = x^1$ is continuous at c, so for the induction step we assume that $g(x) = x^k$ is continuous at c. We want to show that $h(x) = f(x)g(x) = x^{k+1}$ is continuous at c.

$$\lim_{x\to c} h(x) = \lim_{x\to c} f(x)g(x) = f(c)g(c) = c^k c^1 = c^{k+1} = h(c)$$

So it follows that x^k is continuous at c for all $k \in \mathbb{N}$

From presentation problem 6 we also have the limit addition rule. So we can once again use induction to show that $f(x) = \sum_{k=1}^{n} a_k x^k$ is continuous at c. For the base case, let $f(x) = x^k$ and continuous at c, and $h(x) = a_k f(x) = a_k x^k$, then it follows that

$$\lim_{x\to c} h(x) = \lim_{x\to c} a_k f(x) = a_k f(c) = a_k c^k = h(c)$$

So it follows that $f(x) = a_k x^k$ is continuous at c. For the induction step, we assume that $g(x) = \sum_{k=1}^{n} a_k x^k$ is continuous at c, and $f(x) = x^{n+1}$ is continuous at c. And we want to show that $h(x) = \sum_{k=1}^{n+1} a_k x^k$ is continuous at c.

$$\lim_{x\to c} h(x) = \lim_{x\to c} g(x) + a_{n+1} f(x) = g(c) + a_{n+1} f(c) = \sum_{k=1}^{n+1} a_k x^k = h(c)$$

So it follows that $\sum_{k=1}^{n} a_k x^k$ is continuous at c for all $n \in \mathbb{N}$. We then observe that it follows that $p(x)$ is continuous at c. Since c was arbitrary in $\mathbb{R}$, $p(x)$ is continuous at all $\mathbb{R}$

□