Presentation Problems 3

21-355 A

Instructions: Your group should prepare a presentation for the problem corresponding to your group number. After presenting the solution and getting feedback from the class, you have until the beginning of the following class to send a TeX file with a polished form of the solution to the instructor. Make sure all group members' names are in the TeX file. There are 15 points available: 10 for the presentation and 5 for the written proof.

1. Show that $K \subseteq \mathbb{R}$ is compact if and only if K is closed and bounded in $\mathbb{R}$.

2. Let $\{K_n\}_{n=1}^{\infty}$ be a collection of non-empty compact sets in $\mathbb{R}$ such that for each $n \in \mathbb{N}$, $K_n \supseteq K_{n+1}$. Then
$$\bigcap_{n=1}^{\infty} K_n \neq \emptyset.$$

3. Show that any non-empty perfect set in $\mathbb{R}$ is uncountable. Conclude that any non-empty perfect set must contain irrationals.

4. Show that a set $A \subseteq \mathbb{R}$ is connected if and only if $a < c < b$ with $a, b \in A$, then $c \in A$. Use this to show the only connected sets in $\mathbb{R}$ are intervals.

5. Let $f : A \mapsto \mathbb{R}$ and let c be a limit point of A. Then $\lim_{x\to c} f(x) = L$ if and only if for all sequences (x_n) in A such that $x_n \neq c$ for all $n \in \mathbb{N}$ such that $\lim_{n\to\infty} x_n$, $\lim_{n\to\infty} f(x_n) = L$.

6. Let $f, g : A \mapsto \mathbb{R}$ such that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$ for some limit point c of A. Then
   (a) $\lim_{x\to c}[\alpha f(x) + \beta f(x)] = \alpha L + \beta M$ for all $\alpha, \beta \in \mathbb{R}$,
   (b) $\lim_{x\to c}[f(x)g(x)] = LM$, and
   (c) if $M \neq 0$,
   $$\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}.$$

7. Let $A \subseteq \mathbb{R}$ and let $c \in A$. Then $f : A \mapsto \mathbb{R}$ is continuous at c if and only if for all sequences (x_n) in A such that $x_n \to c$, $f(x_n) \to f(c)$.

8. Let $A \subseteq \mathbb{R}$ and $f : A \mapsto \mathbb{R}$. Then f is continuous on A if and only if for all open sets U in $\mathbb{R}$, $f^{-1}(U)$ is open in A. (Recall that a set O is open in A if and only if for all $x \in O$, there is some $r > 0$ such that

$$B_A(x, r) := \{y \in A : |x - y| < r\}$$

is contained in O.

9. Let $A \subseteq \mathbb{R}$ and $f : A \mapsto \mathbb{R}$. Then f is continuous at every isolated point of A.

10. Let A, B be subsets of $\mathbb{R}$ such that $f : A \mapsto \mathbb{R}$, $g : B \mapsto \mathbb{R}$ and $f(A) \subseteq B$. If f is continuous at $c \in A$ and g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.

11. Let $p(x)$ the polynomial

$$p(x) = \sum_{k=1}^{n} a_k x^k.$$

Then p is continuous on $\mathbb{R}$.