Presentation Problems 2–Answers

21-355 A

1. Show that for any $a, b \in \mathbb{R}$ with $a < b$, (a, b) is open in $\mathbb{R}$. Use this to prove that $(-\infty, a)$ and (a, ∞) are open for any $a \in \mathbb{R}$. Conclude that $[a, b]$, $(-\infty, a]$ and $[a, \infty)$ are closed in $\mathbb{R}$.

   Proof. Let $x \in (a, b)$. Then we know that $a < x < b$ and let

   $$r_1 = x - a > 0 \text{ with } a \leq x - r_1 < x$$
   $$r_2 = b - x > 0 \text{ with } x < x + r_2 \leq b$$

   Let $r = \min(r_1, r_2) > 0$, then we have

   $$a \leq x - r_1 \leq x - r < x < x + r \leq x + r_2 \leq b$$

   Now we will show that $B(x, r) \subseteq (a, b)$. Select $0 < \epsilon < r$. Then we know

   $$a \leq x - r < x - \epsilon < x < b, \text{ and}$$
   $$a < x < x + \epsilon < x + r \leq b$$

   Thus, $\forall 0 < \epsilon < r, x \pm \epsilon \in (a, b)$. We have shown that $B(x, r) \subseteq (a, b)$.

   Now, we consider $(-\infty, a)$. Let $x \in (-\infty, a)$, we know x is finite and $x < a$, and $\bigcup_{i=1}^{\infty}(a - i, a) = (\infty, a)$. Since $\forall (a - i, a)$, we have proved that the set is open, and therefore their union is open.

   Similarly $\bigcup_{i=1}^{\infty}(a, a + i) = (a, \infty)$ and $\forall (a, a + i)$, we know the set is open, and thus their union is open.

   For $[a, b]$, we know $[a, b]^c = (-\infty, a) \cup (b, \infty)$. As we have shown $(-\infty, a)$ and (b, ∞) are open, and thus their union is open. Therefore $[a, b]$ is closed.

   For $(-\infty, a]$, we know $(-\infty, a]^c = (a, \infty)$ is open, and therefore $(-\infty, a]$ is closed.

   For $[a, \infty)$, we know $[a, \infty)^c = (-\infty, a)$ is open, and therefore $[a, \infty)$ is closed. □

2. Let $F \subseteq \mathbb{R}$. Then F is closed if and only if every convergent sequence in F converges in F.

   Proof. ($\Rightarrow$) Let F be closed and $(a_n) \to a$ be a convergent sequence in F. Let $\epsilon > 0$; then there exists N such that for each $n \geq N$, $|a_n - a| < \epsilon$ and thus $a_n \in V_\epsilon(a)$, and since $a_n \in F$, $a_n \in V_\epsilon(a) \cap F$. If $a_n = a$, then $a \in F$; otherwise, a is a limit point of F, and since F is closed, $a \in F$. Thus, if F is closed, every convergent sequence in F converges in F.

($\Leftarrow$) Let $F \subseteq \mathbb{R}$ such that every convergent sequence in F converges in F and let a be a limit point of F. Then there exists a sequence (a_n) in F converging to a, implying $a \in F$. Thus, if every convergent sequence in F converges in F, every limit point of F lies in F and F is closed.

Therefore, F is closed if and only if every convergent sequence in F converges in F. □

3. Let $A \subseteq \mathbb{R}$. Then the closure $\overline{A}$ is closed in $\mathbb{R}$.

Proof. let LP_A be the set of limit points of A. So

$$\overline{A} = A \cup LP_A.$$

We wish to show that $\overline{A}$ is closed, so it suffices to show that $\overline{A}^c$ is open.

Let x be an arbitrary and fixed element in $\overline{A}^c$
so $x \notin A, x \notin LP_A$, thus
$\exists \epsilon > 0, \forall y \in A, |x - y| \geq \epsilon$
$\Rightarrow B(x, \epsilon) \cap A = \varnothing$
$\Rightarrow B(x, \epsilon) \subseteq A^c$

If we can also show that $B(x, \epsilon) \subseteq (LP_A)^c$, we've shown that $\overline{A}^c$ is open.
For the sake of contradiction, let's assume $\exists a \in LP_A \cap B(x, \epsilon)$
$a \in LP_A \Rightarrow \forall \epsilon > 0, \exists y \in A$ such that $|a - y| < \epsilon$
$a \in B(x, \epsilon) \Rightarrow \forall \epsilon > 0, \forall y \in A,\ |a - y| \geq \epsilon$
This is a contradiction.
Thus $B(x, \epsilon) \subseteq (LP_A)^c$, we've already shown $B(x, \epsilon) \subseteq A^c$
Therefore $B(x, \epsilon) \subseteq \overline{A}^c$, so $\overline{A}^c$ is open.

□

4. (a) Let F_i be closed for $i = 1, 2, \ldots, N$. Then $\bigcup_{i=1}^{N} F_i$ is closed.

Proof. For any $i = 1, 2, ..., N, F_i$ is closed.
We have proven in lecture that the complement of a closed set must be open, so F_i^c is open for all i. We also proved that the intersection of finite number of open sets is open, thus $\bigcap_{i=1}^{N} F_i^c$ is also open. Since the complement of an open set must be closed, we know $(\bigcap_{i=1}^{N} F_i^c)^c$ is closed.
By De Morgan's Law: $(A \cap B)^c = A^c \cup B^c$.
So $(\bigcap_{i=1}^{N} F_i^c)^c = \bigcup_{i=1}^{N} (F_i^c)^c = \bigcup_{i=1}^{N} F_i$ is closed. □

(b) Let F_i be closed for all i in some indexing set I. Then $\bigcap_{i\in I} F_i$ is closed.

Proof. For any $i \in I, F_i$ is closed.
Since the complement of a closed set must be open, F_i^c is open for all $i \in I$. We have proven in lecture that the union of any number of open sets is open, so $\bigcup_{i\in I} F_i$ is open. Its complement, $(\bigcup_{i\in I} F_i)^c$ is closed.
By De Morgan's Law: $(A \cup B)^c = A^c \cap B^c$.
So $(\bigcup_{i\in I} F_i)^c = \bigcap_{i\in I}(F_i^c)^c = \bigcap_{i\in I} F_i$ is closed. □

5. Let (x_n) be a real sequence such that $\lim_{n\to\infty} x_n = x$. Show that the set $S = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is closed in $\mathbb{R}$.

Proof. Consider the complement of S, we will show that S is closed in $\mathbb{R}$ by showing that S^c is open in $\mathbb{R}$.
Arbitrarily pick $t \in S^c$
Thus, $t \notin \{x_n : n \in \mathbb{N}\}$ and $t \neq x$.
Since $t \neq x$, so we know $\exists r > 0$ st. $|t - x| = r$.
Consider the open ball $B(x, \frac{r}{2})$.
Since the sequence converges to x, from previous homework we know that there are all but infinitely many x_n's in the ball, and there are only finite number of x_n's outside the ball.
Now consider $B(t, \frac{r}{2})$.
It is easy to see that $B(t, \frac{r}{2})$ is disjoint from $B(x, \frac{r}{2})$ since the distance between x and s is r and the ball is an open interval.
So $B(t, \frac{r}{2})$ is disjoint from the ball $B(x, \frac{r}{2})$, which means there are only finite number of x_n's inside the ball $B(t, \frac{r}{2})$.
Now pick the minimum distance δ between t and all those finite number of x_n's.
Pick the minimum of δ and $\frac{r}{2}$, let it be ϵ.
So we have $B(t, \epsilon)$ does not contain any x_n and x.
Therefore $B(t, \epsilon) \subseteq S^c$
Now we just show that $\forall t \in S^c, \exists \epsilon > 0$ st. $B(t, \epsilon) \subseteq S^c$.
Hence S^c open, which means S is closed. □

6. Let $A \subseteq \mathbb{R}$. Show the following are equivalent.

(a) A is dense in $\mathbb{R}$

(b) For any $x \in \mathbb{R}$, there exists some sequence (x_n) in A such that $x_n \to x$.

(c) $\overline{A} = \mathbb{R}$.

Proof. We will show the equivalence by showing (a) $\implies$ (b), (b) $\implies$ (c), and (c) $\implies$ (a).

((a)$\Longrightarrow$(b)) We will find such a sequence via a constructive algorithm (specifically the bisection method). Let A be dense in $\mathbb{R}$ and $x \in \mathbb{R}$ be arbitrary. Define the endpoint $c_n = x - \dfrac{1}{2^n}$ for all $n \in \mathbb{N}$ (notice that it is real). Since A is dense in $\mathbb{R}$, we know there is some $a_n \in A$ such that $c_n \leq a_n \leq x$. I claim $(x_n) = (a_n)$ is the sequence we are looking for.

Let $\varepsilon > 0$ be arbitrary and fixed. See that $|x_n - x| \leq x - \left(x - \dfrac{1}{2^n}\right) = \dfrac{1}{2^n}$ since $x_n = a_n \geq c_n$. Then,

$$\frac{1}{2^n} < \varepsilon \implies \frac{1}{\varepsilon} = 2^n \implies \log_2\left(\frac{1}{\varepsilon}\right) < n \implies n > \log_2\left(\frac{1}{\varepsilon}\right)$$

$$\implies |x - x_n| < \varepsilon \text{ for } n > \log_2\left(\frac{1}{\varepsilon}\right)$$

Let $N = \left\lceil \log_2\left(\frac{1}{\varepsilon}\right) \right\rceil + 1$ and note that $N \in \mathbb{N}$. So for all $n \geq N$ we have $|x - x_n| < \varepsilon$. Since ε was arbitrary, we have shown that (x_n) converges to x. Also, since $x_n = a_n \in A$ for all $n \in \mathbb{N}$, we have $(x_n) \in A$. These are the two essential properties of the sequence and we are done.

((b) $\Longrightarrow$ (c)) Assume for any $x \in \mathbb{R}$, $\exists (x_n) \subseteq A,\ s.t.\ x_n \to x$. Since we want to show $\overline{A} = \mathbb{R}$, equivialently, we would like to prove $\overline{A} \subseteq \mathbb{R}$ and $\overline{A} \supseteq \mathbb{R}$.

($\overline{A} \subseteq \mathbb{R}$)
let $x \in \overline{A}$, then base on our assumption, $x \in \mathbb{R}$, $\overline{A} \subseteq \mathbb{R}$ trivially.

($\overline{A} \supseteq \mathbb{R}$)
let $x \in \mathbb{R}$. Since $\exists (x_n) \subseteq A$, s.t. $x_n \to x$, we know

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n > N,\ |x_n - x| < \epsilon$$

that means $x_n \in B(x, \epsilon)$, and because $x_n \in A$:

$\Rightarrow$ $x_n \in B(x, \epsilon) \cap A$
$\Rightarrow$ x must be a limit point of A,
because we have shown there exists other points in $B(x, \epsilon) \cap A$ that is not$x$
$\Rightarrow$ $x \in \overline{A}$
$\Rightarrow$ $\mathbb{R} \subseteq \overline{A}$

Since $\overline{A} \subseteq \mathbb{R}$ and $\overline{A} \supseteq \mathbb{R}$, that proves $\overline{A} = \mathbb{R}$

((c)$\Longrightarrow$(a)) We will prove this using a proof by contradiction. Assume for sake of contradiction that A is not dense in R. Then there exists a point

$x \in \mathbb{R}$ and $d > 0$ such that $(x-d, x+d) \cap A = \emptyset$ (we will denote this with (*)). Because $\overline{A} = \mathbb{R}$, this implies that $(x-d, x+d) \cap \overline{A} = (x-d, x+d)$. Thus, all points in $(x-d, x+d)$ are limit points of A. Let us consider $y \in (x-d, x+d)$. For any $\epsilon > 0$, $(y-\epsilon, y+\epsilon) \cap A \neq \emptyset$, because y is a limit point of A. Because $(x-d, x+d)$ is an open interval and $y \in (x-d, x+d)$, we can find $e > 0$ such that $(y-e, y+e) \subseteq (x-d, x+d)$. From (*), we know that $(y-e, y+e) \cap A \subseteq (x-d, x+d) \cap A = \emptyset$, thus, $(y-e, y+e) \cap A = \emptyset$. This is a contradiction, as we assumed that $(y-e, y+e) \cap A \neq \emptyset$. Thus, A is dense in $\mathbb{R}$. □

7. Show that $A \subseteq \mathbb{R}$ is open in $\mathbb{R}$ if and only if $A = \operatorname{int} A$.

Proof. ($\Rightarrow$) Assume A is open.
By definition, $A \subseteq \mathbb{R}$ is open if and only if $\forall x \in A$, $\exists r > 0$ such that $B(x, r) \subseteq A$, so every point must be an interior point, so $A \subseteq \operatorname{int} A$. The reverse, $\operatorname{int} A \subseteq A$, is always true since the interior points have to lie in A together with a neighborhood. Thus, $A = \operatorname{int} A$.

($\Leftarrow$) Assume $A = \operatorname{int} A$.
Claim: $\operatorname{int} A$ is an open set.
Let $x \in \operatorname{int} A$. So $\exists r > 0$, such that $B(x, r) \subseteq \operatorname{int} A = A$. Then $\forall x \in \operatorname{int} A = A$, $\exists r > 0$, such that $B(x, r) \subseteq A = \operatorname{int} A$.
Since $\operatorname{int} A$ is an open subset of A and since $A = \operatorname{int} A$, then A must also be open. □

8. Show that $\operatorname{bd} A = \overline{A} \cap \overline{\mathbb{R} \setminus A}$ for any $A \subseteq \mathbb{R}$ and that $\overline{A} = \operatorname{int} A \cup \operatorname{bd} A$.

Proof. We make a generalization for the proof of this statement. We assume that $x \in A$ and $x \notin A^C$. Upon proving the statement with this assumption, we immediately substitute A for A^C in the equation we want to prove in general and immediately realize it holds if $x \in A^C$ as well.

We will first prove that bd $A = \bar{A} \cap \bar{A^C}$ by means of double-containment.

We will first prove that $bd\ A \subseteq \bar{A} \cap \bar{A^C}$. Let $x \in bd\ A$ be arbitrary. Then by the definition of boundary, we know that for all $r > 0$, $B(x, r) \cap A \neq \emptyset$. Since $\bar{A} = A \cup LP(A)$, we know that $x \in A$ gives $x \in \bar{A}$. Note that there exists a $y \in B(x, r)$ such that $y \neq x$, $y \in A^C$, and for all $r > 0$, $(B(x, r) \cap A^C) \setminus \{x\} \neq \emptyset$, also following from $x \in bd\ A$. By the definition of limit point, we conclude $x \in LP(A^C)$, and by the definition of closure, $x \in \bar{A^C}$ as well.

Since $x \in \bar{A}$ and $x \in \bar{A^C}$, we conclude that $x \in \bar{A} \cap \bar{A^C}$, as desired.

We now prove the other direction, that $\bar{A} \cap \bar{A^C} \subseteq bd\ A$. Let $x \in \bar{A} \cap \bar{A^C}$ be arbitrary. Note that $x \in A$, so $x \notin A^C$. Since $x \in \bar{A^C}$, by the definition of closure we conclude that $x \in LP(A^C)$. By the definition of limit point, for all $r > 0$ we see that $(A^C \cap B(x, r))/\{x\} \neq \emptyset$, so $A^C \cap B(x, r) \neq$

$\emptyset$. Moreover, since $x \in A$, it is trivial that $A \cap B(x,r) \neq \emptyset$ (since this intersection contains the point x). By the definition of boundary, we conclude that $x \in \text{bd } A$.

We now proceed to the second part of the problem, which says to prove that $\bar{A} = \text{int } A \cup \text{bd } A$. This will also be done by double-containment. First we will prove that $\bar{A} \subseteq \text{int } A \cup \text{bd } A$. There are two cases to consider:

Case 1: $x \in A$. Choose an $r > 0$. If $B(x,r) \subseteq A$, then $x \in int\ A$. If not, then we have the following:

(a) $x \in A$ and $x \in B(x,r)$, so $B(x,r) \cap A \neq \emptyset$.

(b) There exists a $y \in B(x,r)$ such that y is not in A, so $y \in B(x,r) \cap A^C$ by the definition of set complement. Hence $B(x,r) \cap A^C \neq \emptyset$.

These two claims combined let us conclude that $x \in bd\ A$ by definition of boundary.

Case 2: $x \in LP(A)$. Then by the definition of limit point, for all $r > 0$, $B(x,r) \cap A \neq \emptyset$ (at least x must be contained in it). If $B(x,r) \cap A^C \neq \emptyset$, then $x \in bd\ A$ by definition of boundary. However, if this intersection is empty, then all elements of $B(x,r)$ must instead be in A, so $x \in int\ A$ instead.

Next we will prove that $int\ A \cup bd\ A \subseteq \bar{A}$. Let $x \in int\ A \cup bd\ A$ be arbitrary. Again we will need two cases.

Case 1: If there exists an $r > 0$ such that $B(x,r) \subseteq A$, then by the earlier part of the proof, we know that $x \in A$ by the definition of interior. Since $A \subseteq \bar{A}$, we know that $x \in \bar{A}$.

Case 2: If for all $r > 0$ we have that $B(x,r)$ is not a subset of A, then there must be some element of $B(x,r)$ that is in A^C, so $B(x,r) \cap A^C \neq \emptyset$. Also, $x \in A$ implies $A \cap B(x,r) \neq \emptyset$ since trivially $x \in B(x,r)$. By the definition of boundary point, the existence of these nonempty set intersections lets us conclude that $x \in bd\ A$, as desired.

By the process of double containment, the proof is complete. □

9. For any $A \subseteq \mathbb{R}$, $\mathbb{R}$ is partitioned into the interior A, the exterior of A, and the boundary of A.

   Proof. Show that $\mathbb{R} = \text{int } A \cup \text{ext } A \cup \text{bd } A$ by double containment.
   It is trivial to see that $\text{int } A \cup \text{ext } A \cup \text{bd } A \subseteq \mathbb{R}$ by the definition of interior, exterior and boundary.

   Show that $\mathbb{R} \subseteq \text{int } A \cup \text{ext } A \cup \text{bd } A$.
   let $x \in \mathbb{R}$,
   Case 1: $x \in \text{bd } A$, trivial
   Case 2: $x \notin \text{bd } A$,

$\exists r > 0$ such that $B(x,r) \cap A = \emptyset$ or $B(x,r) \cap A^c = \emptyset$
If $B(x,r) \cap A = \emptyset$ then $x \in \operatorname{ext} A$
If $B(x,r) \cap A^c = \emptyset$ then $B(x,r) \subseteq A$ and $x \in \operatorname{int} A$
Therefore $\mathbb{R} = \operatorname{int} A \cup \operatorname{ext} A \cup \operatorname{bd} A$

Prove that x is in only one of $\operatorname{int} A, \operatorname{ext} A$, and $\operatorname{bd} A$ by contradiction:
Suppose $x \in \operatorname{int} A \cap \operatorname{ext} A$, then $\exists r_1 > 0$ such that $B(x,r_1) \subseteq A$ and $\exists r_2 > 0$ such that $B(x,r_2) \cap A = \emptyset$.
Case 1: $r_1 < r_2$
$B(x,r_1) \subseteq B(x,r_2)$
$B(x,r_1) \cap A = \emptyset$
Contradiction achieved
Case 2: $r_2 < r_1$
$B(x,r_2) \subseteq B(x,r_1) \subseteq A$
Contradiction achieved
Thus, $x \notin \operatorname{int} A \cap \operatorname{ext} A$

Suppose $x \in \operatorname{int} A \cap \operatorname{bd} A$, then $\exists r > 0$ such that $B(x,r) \subseteq A$ and $B(x,r) \cap A^c \neq \emptyset$
Which means $\exists y \in B(x,r) \cap A^c$, which implies that $y \notin A$
Contradiction achieved.

Suppose $x \in \operatorname{ext} A \cap \operatorname{bd} A$, then $\exists r > 0$ such that $B(x,r) \cap A = \emptyset$, and $B(x,r) \cap A \neq \emptyset$.
Contradiction achieved

Since $x \notin \operatorname{int} A \cap \operatorname{ext} A$, $x \notin \operatorname{int} A \cap \operatorname{bd} A$, and $x \notin \operatorname{ext} A \cap \operatorname{bd} A$, for all $A \in \mathbb{R}$, $\mathbb{R}$ is partitioned into the interior of A, the exterior of A, and the boundary of A. □

10. Let $A \subseteq \mathbb{R}$. Then

$$\operatorname{int} A = \bigcup \{U : U \subseteq A \text{ and } U \text{ open in } \mathbb{R}\}$$

and

$$\overline{A} = \bigcap \{F : F \supseteq A \text{ and } F \text{ closed in } \mathbb{R}\}.$$

Proof. We prove the first statement using proof by double containment.

($\subseteq$) Let $x \in \operatorname{int} A$. Note that since $\operatorname{int} A$ is open and $\operatorname{int} A \subseteq A$, then $\operatorname{int} A = U_i$ for some $i \in I$ for an indexing set I of RHS. Then, $x \in U_i \subseteq$ RHS.

($\supseteq$) Let $x \in$ RHS. Then $x \in U_i$ for some $i \in I$. Since U_i is open, we can form an open ball around x such that for an $r > 0$, we have $B(x,r) \subseteq U_i$. Since $U_i \subseteq A$, then $B(x,r) \subseteq A$ so $x \in \operatorname{int} A$.

We now prove the second statement by double containment.

(a) ($\Longrightarrow$)

Let $x \in \overline{A}$. Then x is either $\in A$ or is a limit point of A.

Case $x \in A$.

Then since A is a subset of every set F_i in the RHS' intersection, we know that x is a member of each F_i. Thus x is a member of the intersection, so $x \in$ RHS.

Case x is a limit point of A and $x \notin A$.

Then there exists a sequence (a_n) in A that converges to x. Since A is a subset of every set F_i in the RHS' intersection, (a_n) is in each F_i. Since each F_i is closed, by presentation problem 2 we know that (a_n) converges in F_i, so $x \in$ each F_i. Thus x is a member of the intersection, so $x \in$ RHS.

Thus $\overline{A} \subseteq$ RHS.

(b) ($\Longleftarrow$)

Let $x \in$ RHS. Then we know that x is a member of every F_i in the RHS' intersection.

Now $\overline{A} \supseteq A$ (since every element of A is in the closure), and by presentation problem 3 we know $\overline{A}$ is closed in $\mathbb{R}$. Thus $\overline{A}$ is actually one of the F_i's in the RHS' intersection. Since x is a member of every F_i, it is a member of $\overline{A}$.

Thus RHS $\subseteq \overline{A}$.

Thus by double containment, $\overline{A} = \bigcap\{F : F \supseteq A \text{ and } F \text{ closed in } \mathbb{R}\}$. $\square$

11. Show that the only sets both open and closed in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$.

Hint: Let $a_1 \in A$ and $b_1 \in A^c$, assuming without loss of generality that $a_1 < b_1$ (why is this possible?). Let c_1 be the midpoint of a_1 and b_1. If $c_1 \in A$, let $a_2 = c_1$ and $b_2 = b_1$; otherwise, let $a_2 = a_1$ and $b_2 = c_1$. Show this construction can be continued inductively. Find $x \in \bigcap_{n=1}^{\infty}[a_n, b_n]$ and show that $a_n \to x$ and $b_n \to x$.

Proof. To prove that there the only sets that are both open and closed in $\mathbb{R}$ are the $\emptyset$ and $\mathbb{R}$ we will do a proof by contradiction.

Assume that there is a set A that is both open and closed that is not $\emptyset$ or $\mathbb{R}$. Notice that A open $\Rightarrow A^c$ closed, and A closed $\Rightarrow A^c$ open. Let $a_1 \in A$ and $b_1 \in A^c$, assuming without loss of generality that $a_1 < b_1$ which is possible because if $a_1 > b_1$ we can just relabel A to be A^c. Define a point c_1 to be the midpoint of a_1 and b_1 $(\frac{a_1+b_1}{2})$. If $c_1 \in A$ then let $a_2 = c_1$

and $b_2 = b_1$, otherwise let $a_2 = a_1$ and $b_2 = c_1$. This construction can be continued inductively. Assume that we have a_k and b_k in A and A^c respectively. We can find c_k via $\frac{a_k+b_k}{2}$, and since A and A^c partition $\mathbb{R}$ that means $c_k \in A$ or $c_k \in A^c$. If $c_k \in A$ then $a_{k+1} = c_k$ and $b_{k+1} = b_k$, else $a_{k+1} = a_k$ and $b_{k+1} = c_k$.

Since $[a_1, b_1] \supseteq [a_2, b_2] \supseteq, ..., \supseteq [a_k, b_k]$, then by the nested interval property $\exists x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$.

$(a_n) \to x$ and $(b_n) \to x$. Let d be the length of the interval which is $|b_1 - a_1|$. From the nested interval property we know that $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$. Fix $\epsilon > 0$, and let $N \in \mathbb{N}$ s.t $N > (d/\epsilon) + 1$. Assume $n \geq N$. Then:

$$|a_n - x| \leq \frac{d}{2^{n-1}} \leq \frac{d}{n-1} < \epsilon$$

and:

$$|b_n - x| \leq \frac{d}{2^{n-1}} \leq \frac{d}{n-1} < \epsilon$$

Since A is closed $\Rightarrow$ A includes all of its limit points, so $x \in A$, however since A^c is closed $\Rightarrow$ A^c includes all of its limit points so $x \in A^c$. This is a contradiction however because A and A^c partition $\mathbb{R}$ which means that $x \in A \Rightarrow x \notin A^c$. Since we have reached a contradiction this means that our original assumption is false and there is no sets other than $\emptyset$ and $\mathbb{R}$ that are both open and closed in $\mathbb{R}$.

□