Presentation Problems 2

21-355 A

Instructions: Your group should prepare a presentation for the problem corresponding to your group number. After presenting the solution and getting feedback from the class, you have until the beginning of the following class to send a Tex file with a polished form of the solution to the instructor. Make sure all group members' names are in the Tex file. There are 15 points available: 10 for the presentation and 5 for the written proof.

1. Show that for any $a, b \in \mathbb{R}$ with $a < b$, (a, b) is open in $\mathbb{R}$. Use this to prove that $(-\infty, a)$ and (a, ∞) are open for any $a \in \mathbb{R}$. Conclude that $[a, b]$, $(-\infty, a]$ and $[a, \infty)$ are closed in $\mathbb{R}$.

2. Let $F \subseteq \mathbb{R}$. Then F is closed if and only if every convergent sequence in F converges in F.

3. Let $A \subseteq \mathbb{R}$. Then the closure $\overline{A}$ is closed in $\mathbb{R}$.

4. (a) Let F_i be closed for $i = 1, 2, \ldots, N$. Then $\bigcup_{i=1}^{N} F_i$ is closed.
   (b) Let F_i be closed for all i in some indexing set I. Then $\bigcap_{i \in I} F_i$ is closed.

5. Let (x_n) be a real sequence such that $\lim_{n\to\infty} x_n = x$. Show that the set $S = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ is closed in $\mathbb{R}$.

6. Let $A \subseteq \mathbb{R}$. Show the following are equivalent.
   (a) A is dense in $\mathbb{R}$
   (b) For any $x \in \mathbb{R}$, there exists some sequence (x_n) in A such that $x_n \to x$.
   (c) $\overline{A} = \mathbb{R}$.

7. Show that $A \subseteq \mathbb{R}$ is open in $\mathbb{R}$ if and only if $A = \operatorname{int} A$.

8. Show that $\operatorname{bd} A = \overline{A} \cap \overline{\mathbb{R} \setminus A}$ for any $A \subseteq \mathbb{R}$ and that $\overline{A} = \operatorname{int} A \cup \operatorname{bd} A$.

9. For any $A \subseteq \mathbb{R}$, $\mathbb{R}$ is partitioned into the interior A, the exterior of A, and the boundary of A.

10. Let $A \subseteq \mathbb{R}$. Then

$$\operatorname{int} A = \bigcup \{U : U \subseteq A \text{ and } U \text{ open in } \mathbb{R}\}$$

and

$$\overline{A} = \bigcap \{F : F \supseteq A \text{ and } F \text{ closed in } \mathbb{R}\}.$$

11. Show that the only sets both open and closed in $\mathbb{R}$ are $\emptyset$ and $\mathbb{R}$.

    Hint: Let $a_1 \in A$ and $b_1 \in A^c$, assuming without loss of generality that $a_1 < b_1$ (why is this possible?). Let c_1 be the midpoint of a_1 and b_1. If $c_1 \in A$, let $a_2 = c_1$ and $b_2 = b_1$; otherwise, let $a_2 = a_1$ and $b_2 = c_1$. Show this construction can be continued inductively. Find $x \in \bigcap_{n=1}^{\infty} [a_n, b_n]$ and show that $a_n \to x$ and $b_n \to x$.