Presentation Problems 1–Proofs

21-355 A

1. Let (a_n) be a convergent sequence. Then (a_n) is bounded. In addition, let (a_{n_k}) be a subsequence of (a_n). Then the subsequence (a_{n_k}) converges to $\lim a_n$.

Theorem 1. *Let (a_n) be a convergent sequence. Then (a_n) is bounded.*

Proof. From the definition of a convergent sequence we know:

(a_n) is convergent $\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ it follows that $|a_n - l| < \epsilon$

In order for (a_n) to be bounded we must show:

$$\exists M > 0 \text{ s.t. } |a_n| \leq M, \forall n \in \mathbb{N}$$

We know from the convergence of (a_n) that $|a_n - l| < \epsilon$, so by the definition of absolute value it follows that $-\epsilon < a_n - l < \epsilon$. From this it follows that $-\epsilon + l < a_n < \epsilon + l$. While this is correct we don't know that l is positive so we will use $-\epsilon - |l| < a_n < \epsilon + |l|$ so we can be certain that our lower bound is the opposite of our upper bound, which would mean we could rewrite the statement as $|a_n| < \epsilon + |l|$.

We now have a value, $\epsilon + |l|$, that we can use for M that we know bounds the sequence when $n \geq N$. However we cannot be certain that this value of M will bound the sequence when $n < N$, so we should let $M = \max(|a_1|, |a_2|, ..., |a_{n-2}|, \epsilon + |l|)$.

Note that because every term in the sequence up to a_{n-1} is less than or equal to the greatest term of that part of the sequence, and every term from a_n onward it follows that $|a_n| \leq M \ \forall n \in \mathbb{N}$. □

Theorem 2. *Let (a_{n_k}) be a sub-sequence of (a_n). The the sub-sequence (a_{n_k}) converges to the same limit as (a_n).*

Proof. Aside: Observe that $n_1 \geq 1$ because as part of the definition of a sub-sequence the first term of the sub-sequence must be at least the first term of the original sequence. Assume that $n_k \geq k$ then it follows that:

$$n_k + 1 \geq k + 1$$

$$n_{k+1} \geq n_k + 1$$

Because the $k+1$ term of the sub-sequence must be at least the next term of the sequence.

$$n_{k+1} \geq n_k + 1 \geq k + 1 \Rightarrow n_{k+1} \geq k + 1$$

From the definition of a convergent sequence we know:

(a_n) is convergent $\Rightarrow \forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ it follows that $|a_n - l| < \epsilon$

Since $n_k \geq k$, let $k \in \mathbb{N}$ s.t. $k \geq N$. This implies that $n_k \geq N$, so it follows that $|a_{n_k} - l| < \epsilon$. So the sub-sequence converges to the same value as the original sequence.

$\square$

2. Let (a_n) and (b_n) be sequences such that $\lim a_n = a$ and $\lim b_n = b$. Then for any $\alpha, \beta \in \mathbb{R}$, $\lim(\alpha a_n + \beta b_n) = \alpha a + \beta b$ and $\lim(a_n b_n) = ab$. Further, $\lim \frac{a_n}{b_n} = \frac{a}{b}$ provided $b \neq 0$.

   Proposition 3. *Let (a_n) and (b_n) be real sequences such that $a_n \to a$ and $b_n \to b$ and let $\alpha, \beta \in \mathbb{R}$. Then $\lim(\alpha a_n + \beta b_n) = \alpha a + \beta b$.*

   Proof. We first prove that $\lim(a_n + b_n) = a + b$. Equivalently, we want to show that for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for $n > N$, $|a_n + b_n - (a+b)| < \epsilon$.
   Let $\epsilon > 0$. Then there is some $N_1 \in \mathbb{N}$ such that if $n \geq N_1, |a_n - a| < \frac{\epsilon}{2}$.
   Similarly, there is some $N_2 \in \mathbb{N}$ such that if $n \geq N_2, |b_n - b| < \frac{\epsilon}{2}$.
   Pick $N = \max(N_1, N_2)$, then for $n \geq N$,

$$\begin{aligned} |a_n + b_n - (a+b)| &= |(a_n - a) + (b_n - b)| \\ &\leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

   Therefore, for $n > N$, $|a_n + b_n - (a+b)| < \epsilon$ holds for all $\epsilon > 0$.
   So $\lim(a_n + b_n) = a + b$

   We then prove that $\lim(\alpha a_n) = \alpha a$ for any $\alpha \in \mathbb{R}$. Equivalently, we want to show that for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for $n \geq N$, $|\alpha a_n - \alpha a| < \epsilon$.

   - Case 1: $\alpha \neq 0$. Since $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for $n \geq N$, we have $|a_n - a| < \frac{\epsilon}{|\alpha|}$. By properties of absolute value, $|\alpha a_n - \alpha a| < \epsilon$ follows.
   - Case 2: $\alpha = 0$. Then $|\alpha a_n - \alpha a| = 0 < \epsilon$

Since $\lim(a_n + b_n) = a + b$ and $\lim(\alpha a_n) = \alpha a$, $\lim(\alpha a_n + \beta b_n) = \alpha a + \beta b$ follows immediately.

□

Proposition 4. *Let* (a_n) *and* (b_n) *be real sequences such that* $a_n \to a$ *and* $b_n \to b$*. Then* $\lim(a_n b_n) = ab$*.*

Proof. Equivalently, we want to show that for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for $n > N$, $|a_n b_n - ab| < \epsilon$.
Since sequence (b_n) converges to limit b, it is bounded. Let K be the bound, $|b_n| < K$. Since $\epsilon > 0$, there is some $N_1 \in \mathbb{N}$ such that for $n \geq N_1$, we have $|a_n - a| < \frac{\epsilon}{2(K+1)}$. Similarly, there is some $N_2 \in \mathbb{N}$ such that if $n \geq N_2$, $|b_n - b| < \frac{\epsilon}{2(|a|+1)}$.
Pick $N = \max(N_1, N_2)$, then for $n \geq N$,

$$\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\
&\leq |(a_n - a)b_n| + |a(b_n - b)| \\
&< \frac{\epsilon}{2(K+1)} \times K + |a| \times \frac{\epsilon}{2(|a|+1)} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Therefore, $\lim(a_n b_n) = ab$ holds.

□

Proposition 5. *Let* (a_n) *and* (b_n) *be real sequences such that* $a_n \to a$ *and* $b_n \to b \neq 0$*. Then* $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$*.*

Proof. We have shown that $\lim(a_n b_n) = ab$. If we can prove that $\lim\left(\frac{1}{b_n}\right) = \frac{1}{b}$, then $\lim(\frac{a_n}{b_n}) = \frac{a}{b}$ follows immediately. Proving $\lim\left(\frac{1}{b_n}\right) = \frac{1}{b}$ is equivalent to proving that for any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for $n > N$, $|\frac{1}{b_n} - \frac{1}{b}| < \epsilon$.
Suppose $b > 0$. Since $\epsilon > 0$, there is some $N_1 \in \mathbb{N}$ such that for $n \geq N_1$, $|b_n - b| < \frac{b^2}{2}\epsilon$. Besides, there is some $N_2 \in \mathbb{N}$ such that for $n \geq N_2$, $|b_n - b| < \frac{b}{2}$. By properties of absolute value, we have that $\frac{b}{2} < b_n < \frac{3b}{2}$. Since we have supposed that $b > 0$, this implies that $|b_n| > \frac{b}{2}$.
Pick $N = \max(N_1, N_2)$, then for $n > N$,

$$\begin{aligned}
\left|\frac{1}{b_n} - \frac{1}{b}\right| &= \left|\frac{b - b_n}{bb_n}\right| \\
&< \frac{\frac{b^2}{2}\epsilon}{b \times \frac{b}{2}} \\
&= \epsilon
\end{aligned}$$

Suppose $b < 0$. Since $\lim(\alpha a_n) = \alpha a$, we have $\lim(-1 \times b_n) = -b$, where $-b > 0$. Then by the above proof, we have $\lim\left(-\frac{1}{b_n}\right) = -\frac{1}{b}$. Again using $\lim(\alpha a_n) = \alpha a$, we have $\lim\left(-1 \times -\frac{1}{b_n}\right) = \lim\left(\frac{1}{b_n}\right) = -1 \times -\frac{1}{b} = \frac{1}{b}$.

Therefore, $\lim\left(\frac{1}{b_n}\right) = \frac{1}{b}$ holds. Since $\lim(a_n b_n) = ab$, $\lim\left(\frac{a_n}{b_n}\right) = \frac{a}{b}$ holds provided that $b \neq 0$. □

3. Let (a_n) and (b_n) be sequences such that $\lim a_n = a$ and $\lim b_n = b$.

   (a) If $a_n \geq \alpha$ for all $n \in \mathbb{N}$, then $a \geq \alpha$. Similarly, if $a_n \leq \beta$ for all $n \in \mathbb{N}$, then $a \leq \beta$.

   Proof. Assume for sake of contradiction that $a < \alpha$. By definition of a limit of a sequence, for any arbitrary $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, we have that $|a_n - a| < \epsilon$. Fix an arbitrary $\epsilon > 0$. Then we have $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \epsilon$. We now want to show that $|a_n - \alpha| < \epsilon$. This implies that α is a limit of (a_n), and by the uniqueness of limits, then $a = \alpha$ and we would have a contradiction. By the assumption, $a < \alpha \Longrightarrow a_n - a > a_n - \alpha$. Since $a_n \geq \alpha$, then $a_n - \alpha \geq 0$ and so $|a_n - a| \geq |a_n - \alpha| > 0$. However, since $|a_n - a| < \epsilon$, this means that $|a_n - \alpha| < \epsilon$. Since ϵ was arbitrary, this proves that α is a limit of (a_n), and we are done.

   A similar proof follows for the case where $a_n \leq \beta$ for all $n \in \mathbb{B}$ by assuming that $a > \beta$. By defintion of a limit of a sequence, we have for any arbitrary ϵ, there exists $N \in \mathbb{N}$ such that for all $n > N$, $|a_n - a| < \epsilon$. By the assumption, $a - a_n > \beta - a_n$. Also, since $\beta - a_n \geq 0$, then $a - a_n > \beta - a_n \geq 0 \Longrightarrow |a - a_n| > |\beta - a_n|$ and thus $\epsilon > |a_n - a| = |a - a_n| > |\beta - a_n| = |a_n - \beta| \geq 0 \Longrightarrow |a_n - \beta| < \epsilon$. Then β is a limit of (a_n), thus $\beta = a$, and we have a contradiction. □

   (b) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

   Proof. Assume for sake of contradiction that $a > b$. Then b cannot be the limit of (a_n) because limits are unique.

   By negation of the definition of limit of a sequence, there exists ϵ such that for all $N \in \mathbb{N}$ there exists $n > N$ such that $|a_n - b| > \epsilon$. Let us choose this ϵ.

   By definition of limits of a sequence, there exists $N_1 \in \mathbb{N}$ such that for all $n > N_1$, $|a_n - a| < \epsilon$

   Similarly, there exists $N_2 \in \mathbb{N}$ such that for all $n > N_1$, $|b_n - b| < \epsilon$

   Let $N = N_1 + N_2$ and let $n > N$ be arbitrary and fixed. Then we have the following five inequalities:

$$
\begin{aligned}
&(1) \quad |a_n - b| > \epsilon \\
&(2) \quad |a_n - a| < \epsilon \\
&(3) \quad |b_n - b| < \epsilon \\
&(4) \quad a_n \leq b_n \\
&(5) \quad a > b
\end{aligned}
$$

Given our choice of n, there are 2 cases: either $b \geq a_n$ or $b < a_n$.

i. If $b \geq a_n$, then
$\epsilon < |a_n - b|$ [by (1)]
$= b - a_n$ [since $b \geq a_n$ for this case]
$< a - a_n$ [since $a > b$ by (5)]
$= |a - a_n|$ [since $a > b \geq a_n$].
This is a contradiction because by (2) we have that $|a_n - a| < \epsilon$

ii. If $b < a_n$, then
$\epsilon < |a_n - b|$ [by (1)]
$= a_n - b$ [since $a_n > b$ for this case]
$\leq b_n - b$ [since $b_n \geq a_n$ by (4)]
$= |b_n - b|$ [since $b_n \geq a_n \geq b$].
This is a contradiction because by (3) we have that $|b_n - a| < \epsilon$.

□

4. Let (a_n), (b_n), and (c_n) be sequences in $\mathbb{R}$ such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $\lim a_n = \lim c_n = \gamma$, then $\lim b_n = \gamma$. **Note:** It is NOT given that (b_n) converges.

Proof. Suppose $\lim a_n = \lim c_n = \gamma$.
Then for all $\epsilon > 0$ and $n \geq N$ for some $N = \max\{N_1, N_2\}, N_1, N_2 \in \mathbb{N}$
$|a_n - \gamma| < \epsilon$, and $|c_n - \gamma| < \epsilon$.
=> $-\epsilon < a_n - \gamma < \epsilon$ and $-\epsilon < c_n - \gamma < \epsilon$
=> $-\epsilon + \gamma < a_n < \gamma + \epsilon$ and $-\epsilon + \gamma < c_n < \gamma + \epsilon$
Since $b_n \geq a_n$, $-\epsilon + \gamma < a_n \leq b_n$
and since $b_n \leq c_n$, $b_n \leq c_n < \epsilon + \gamma$
Thus $-\epsilon + \gamma < b_n < \epsilon + \gamma$
=> $|b_n - \gamma| < \epsilon$
Therefore $\lim b_n = \gamma$

□

5. Prove that an increasing, bounded sequence (a_n) converges to $\sup\{a_n\}$ and a decreasing, bounded sequence (b_n) converges to $\inf\{b_n\}$. Show that for any bounded sequence (a_n), the sequences (y_n) and (z_n) where

$$y_n := \sup\{a_k : k \geq n\}$$

and

$$z_n := \inf\{a_k : k \geq n\}$$

converge. (These limits are defined as the limit superior, $\limsup a_n$, and the limit inferior, $\liminf a_n$, respectively. Thus, any bounded sequence has a limit superior and limit inferior.)

Theorem 6. *Let (a_n) be an increasing, bounded sequence and (b_n) be a decreasing, bounded sequence. Then $a_n \to \sup\{a_n\}$ and $b_n \to \inf\{b_n\}$.*

Proof. Let s be the least upper bound of the sequence (a_n). so $s = \sup\{a_n\}$
We want to show that $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, \mid a_n - s \mid < \epsilon$.
It is equivalent to show that $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N, s - \epsilon < a_n < s + \epsilon$
Since s is the supremum, $s + \epsilon$ is an upper bound of (a_n). So $a_n < s + \epsilon \ \forall a_n \in (a_n)$
Since s is the supremum, $s - \epsilon$ must not be an upper bound of (a_n). So $\exists N \in \mathbb{N},\ \forall n \geqslant N,\ a_n > s - \epsilon$
We've shown $\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geqslant N,\ s - \epsilon < a_n < s + \epsilon$
Thus the supremum of (a_n) is also the limit of the sequence. Hence, an increasing, bounded sequence (a_n) converges to sup $\{a_n\}$

Similarly, we can prove that a decreasing, bounded sequence (b_n) converges to inf $\{b_n\}$. □

Theorem 7. *Let (a_n) be a bounded sequence in $\mathbb{R}$. Then $\limsup a_n$ and $\liminf a_n$ exist.*

Proof. By the definition of the sequence (y_n), the sequence continuously takes the largest element from part of (a_n). Hence, (y_n) must be a decreasing sequence. Since (a_n), (y_n) is bounded as well. Since (y_n) is a decreasing, bounded sequence, by the first part of the problem, we know that (y_n) converges to its infimum.

Similarly, we can show that (z_n) is an increasing, bounded sequence, and it converges to its supremum. □

6. Prove that for any bounded sequence (a_n), $\liminf a_n \leq \limsup a_n$.

Proof. Assume (a_n) is bounded. Let $p_n := \inf\{a_k : k \geq n\}$ and let $q_n := \sup\{a_k : k \geq n\}$.
So $\liminf a_n = \lim p_n := x$ and $\limsup a_n = \lim q_n := y$.
Assume for sake of contradiction that $x > y$.
So $\exists \delta_1 \in \mathbb{N}$ s.t. $n \geq \delta_1 \implies |p_n - x| < \frac{|x-y|}{2}$
and $\exists \delta_2 \in \mathbb{N}$ s.t. $n \geq \delta_2 \implies |q_n - x| < \frac{|x-y|}{2}$
Without loss of generality, assume $\delta_1 \geq \delta_2$.
We now have $\frac{y-x}{2} = \frac{-|x-y|}{2} < p_{\delta_1} - x \implies \frac{y+x}{2} < p_{\delta_1}$
and $q_{\delta_1} - y < \frac{|x-y|}{2} = \frac{x-y}{2} \implies q_{\delta_1} < \frac{x+y}{2}$

and so $q_{\delta_1} < \frac{x+y}{2} < p_{\delta_1} \implies q_{\delta_1} < p_{\delta_1}$.
This means that the infimum of a set is larger than the supremum, which is a contradiction since the infimum is a lower bound and the supremum is an upper bound.
Therefore $\liminf a_n \leq \limsup a_n$. □

Show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists.

Proof. ($\implies$) Assume $\liminf a_n = \limsup a_n := L$.
Let $I_n = \inf\{a_k : k \geq n\}$
So $\forall \varepsilon > 0, \exists q \in \mathbb{N}$ s.t. $n \geq q \implies |I_n - L| < \varepsilon$
We know $I_q \leq a_i \, \forall i \geq q$ and $|I_q - L| < \varepsilon \implies -\varepsilon < I_q - L$.
So $\forall i \geq q$, $a_i - L > -\varepsilon$.

Now let $S_n = \sup\{a_k : k \geq n\}$.
We know $\forall \varepsilon > 0, \exists p \in \mathbb{N}$ s.t. $n \geq p \implies |S_n - L| < \varepsilon$.
We also know $S_p \geq a_i \, \forall i \geq p$ and $|S_p - L| < \varepsilon \implies S_p - L < \varepsilon$.
So $\forall i \geq p$, $a_i - L < \varepsilon$.
Therefore, $\forall i \geq \max(p, q)$, $-\varepsilon < a_i - L < \varepsilon$
and so $\forall i \geq \max(p, q)$, $|a_i - L| < \varepsilon$ as desired.

($\Leftarrow$) Assume $\lim a_n$ exists, then want to show that $\limsup a_n = \liminf a_n$.
First, we know that a_n is bounded and converges, so it has $\limsup a_n$ and $\liminf\ a_n$.
Now WTS $\limsup a_n = \lim a_n = L$
By the definition of limit, we know that $\exists N_2 \in \mathbb{N}$ s.t. $|a_k - L| < \varepsilon/2 \; \forall k \geq N_2$.
$\limsup a_n = \lim S_n$ where $S_n = \sup\{a_k : k \geq n\}$

From Lemma 1.3.8, we know that S_{N_2} is the supremum iff
$\forall \varepsilon > 0, \exists a_x \in \{a_k : k \geq N_2\}$ such that $|S_{N_2} - a_x| < \varepsilon/2$
$|S_{N_2} - L| = |S_{N_2} - a_k + a_k - L| \leq |S_{N_2} - a_k| + |a_k - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon \; \forall k \geq N_2$
So $\limsup a_n = L$
$\liminf a_n = \lim I_n$ where $I_n = \inf\{a_k : k \geq n\}$
By the same reasoning as in the last part, we have
$\forall \varepsilon > 0, \exists a_x \in \{a_k : k \geq N_2\}$ such that $|I_{N_2} - a_x| < \varepsilon/2$
$|I_{N_2} - L| = |I_{N_2} - a_k + a_k - L| \leq |I_{N_2} - a_k| + |a_k - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon \; \forall k \geq N_2$
So $\liminf a_n = L$, and so $\limsup a_n = \liminf a_n$ □

7. A Cauchy sequence is bounded and a convergent sequence is Cauchy.

Proof. Let (a_n) be a Cauchy sequence. Let $\varepsilon > 0$. By definition of Cauchy sequence, we know that there exists some $N \in \mathbb{N}$ such that if $m, n \geq N$ then

$$|a_m - a_n| < \varepsilon.$$

By the properties of absolute value and since $\varepsilon > 0$, we know that

$$-\varepsilon < a_n - a_m < \varepsilon.$$

We take $m = N$, giving us

$$-\varepsilon < a_n - a_N < \varepsilon.$$

Adding a_N to both sides this gives us the inequality

$$-\varepsilon + a_N < a_n < \varepsilon + a_N.$$

Now we have that for all $n \geq N$

$$\begin{aligned} &a_n < \varepsilon + a_N, \\ &a_n > -\varepsilon + a_N. \end{aligned}$$

Notice that this gives us an upper and lower bound for all the elements in (a_n) after some finite index N. This means that if we take the maximum and minimum over the elements before index N (with the above bounds included), we can derive an upper and lower bound for all elements in (a_n). It follows that for all $n \in \mathbb{N}$, we have that

$$\begin{aligned} &a_n \leq \max\{a_1, a_2, \ldots, a_{N-1}, \ \ \varepsilon + a_N\}, \\ &a_n \geq \min\{a_1, a_2, \ldots, a_{N-1}, -\varepsilon + a_N\}. \end{aligned}$$

Thus (a_n) is bounded.

Let (a_n) be a convergent sequence and L be its limit. Recall that by definition of limits, we know that for all $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that if $n \geq N$ then

$$|a_n - L| < \varepsilon.$$

Let $\varepsilon > 0$. Since $\frac{\varepsilon}{2} > 0$, we know by definition of limits that there exists some $N \in \mathbb{N}$ such that for all $n \geq N$ we have that

$$|a_n - L| < \frac{\varepsilon}{2},$$

We will now show that (a_n) is Cauchy. Let $m, n \geq N$. Notice that it is enough to show that

$$|a_m - a_n| < \varepsilon.$$

By the triangle inequality, we have that

$$\begin{aligned} |a_m - a_n| &\leq |a_m - L| + |L - a_n| \\ &= |a_m - L| + |a_n - L| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Therefore we have shown that $|a_m - a_n| < \varepsilon$, so it follows that (a_n) is Cauchy. $\square$

8. Let $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} \alpha a_k + \beta b_k = \alpha A + \beta B.$$

Proof. For any $m \in \mathbb{N}$, we denote the following partial sums:

$$s_m = \sum_{i=1}^{m} b_i$$
$$t_m = \sum_{i=1}^{m} r_i$$

We also denote the corresponding infinite series as

$$A = \sum_{i=1}^{\infty} a_i$$
$$B = \sum_{i=1}^{\infty} b_i$$

Finally, we let α and β be arbitrary real constants. Then by the definition of an infinite series, we know that

$$A = \lim(t_m)$$
$$B = \lim(s_m)$$

Multiplying both sides of both equations by constants gives

$$\alpha A = \alpha \lim(t_m)$$
$$\beta B = \beta \lim(s_m)$$

We know algebraic properties of limits from Problem 2 of this set of presentations. For now we utilize the scalar multiple property to conclude

$$\alpha A = \alpha \lim(t_m) = \lim(\alpha t_m)$$
$$\beta B = \beta \lim(s_m) = \lim(\beta s_m)$$

Then, adding these two equations together,

$$\alpha A + \beta B = \lim(\alpha t_m) + \lim(\beta s_m)$$

Next, we can use the additive property of limits from Problem 2 to conclude that

$$\alpha A + \beta B = \lim(\alpha t_m + \beta s_m)$$

Finally, we conclude by the definition of an infinite series that

$$\alpha A + \beta B = \sum_{i=1}^{\infty}(\alpha a_i + \beta b_i),$$

as desired. □

9. The series $\sum_{k=1}^{\infty} x_k$ converges if and only if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > m \geq N$,

$$\left|\sum_{k=m+1}^{n} x_k\right| < \varepsilon.$$

Conclude that if $\sum_{k=1}^{\infty} x_k$ converges, then $\lim_{k\to\infty} x_k = 0$.

Proof. ($\Rightarrow$) Let $L \in \mathbb{R}$ be the limit of the series $\sum_{k=1}^{\infty} x_k$. Then we know that the sequence of partial sums (S_n) is convergent. As we have proved that a convergent series is Cauchy. $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, such that for any $n > m \geq N$, we have

$$|S_n - S_m| < \epsilon.$$

WLOG let $n \geq m$ and we can rewrite the inequality above as

$$\left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k\right| = \left|\sum_{k=m+1}^{n} x_k\right| < \epsilon.$$

($\Longleftarrow$) We can rewrite $\left|\sum_{k=m+1}^{n} x_k\right|$ as

$$\left|\sum_{k=m+1}^{n} x_k\right| < \left|\sum_{k=1}^{n} x_k - \sum_{k=1}^{m} x_k\right| < \epsilon$$

the last inequality following from the fact that we can choose m, n larger than some N. Thus, we know the sequence of partial sums is Cauchy, which implies the convergence of this series.

Since $\sum_{k=1}^{\infty} x_k$ converges, we know for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > m \geq N$,

$$\left|\sum_{k=m+1}^{n} x_k\right| < \epsilon.$$

Let $m = n - 1$ and this inequality yields

$$\left|\sum_{k=m+1}^{n} x_k\right| = \left|\sum_{n}^{n} x_k\right| = |x_n| < \epsilon$$

Since this is true for all $n > N$, it must follow that $(x_n) \to 0$ as $n \to \infty$

□

10. If $\sum_{k=1}^{\infty} |x_k|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges.

Proof. From the result of 9, it is enough to show that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > m \geq N$,

$$\left|\sum_{k=m+1}^{n} x_k\right| < \epsilon$$

Let $\epsilon > 0$. Then by using the result from 9 along with the fact that $\sum_{k=1}^{\infty} |x_k|$ converges, we know that there exists $N \in \mathbb{N}$ such that for any $n > m \geq N$,

$$\left|\sum_{k=m+1}^{n} |x_k|\right| < \epsilon$$

Now, let $n > m \geq N$ be arbitrary. Then we know that for $m+1 \leq k \leq n$, by the properties of absolute value

$$-|x_k| \leq x_k \leq |x_k|$$

which tells us that

$$-\sum_{k=m+1}^{n} |x_k| = \sum_{k=m+1}^{n} -|x_k| \leq \sum_{k=m+1}^{n} x_k \leq \sum_{k=m+1}^{n} |x_k|$$

We can then conclude that, by the properties of absolute value,

$$\left|\sum_{k=m+1}^{n} x_k\right| \leq \left|\sum_{k=m+1}^{n} |x_k|\right| < \epsilon$$

So we have found $N \in \mathbb{N}$ such that for all $n > m \geq N$,

$$\left|\sum_{k=m+1}^{n} x_k\right| < \epsilon$$

Thus by the result of 9, we know that $\sum_{k=1}^{\infty} x_k$ converges. □

11. Let (x_n) be a decreasing sequence such that $\lim x_n = 0$. Then $\sum_{k=1}^{\infty}(-1)^{k+1}x_k$ converges.

Proof. Let $S_n = \sum_{k=1}^{n}(-1)^{k+1}x_k$.

First, we show $x_i \geq 0$ for all i. If $x_i < 0$ for some i, choose $\epsilon = -\frac{x_i}{2}$; since (x_n) is decreasing, for all $j \geq i$, $x_j < 0$ and $|x_j| = -x_j \geq -x_i > \epsilon$, so (x_n) does not converge. Thus $x_i \geq 0$ for all i.

For all $\epsilon > 0$, there exists N such that $\forall n \geq N$, $x_i \leq |x_i| < \epsilon$.

We show (x_n) to be Cauchy. Let $m, n \geq N$ and without loss of generality, $m \leq n$. There are four cases:

- **m is even, n is odd.** Since (x_n) is nonnegative and decreasing, $S_n - S_m = x_{m+1} - x_{m+2} + \cdots + x_n = (x_{m+1} - x_{m+2}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq x_n \geq 0$ and $S_n - S_m = x_{m+1} + (-x_{m+2} + x_{m+3}) + ... + (-x_{n-1} + x_n) \leq x_{m+1} < \epsilon$. Thus $|S_n - S_m| < \epsilon$.
- **m is even, n is even.** Then $S_n - S_m = x_{m+1} - x_{m+2} + \cdots - x_n = x_{m+1} + (-x_{m+2} + x_{m+3}) + \cdots + (-x_{n-2} + x_{n+1}) - x_n \leq x_{m+1} - x_n \leq x_{m+1} < \epsilon$. Thus $|S_n - S_m| < \epsilon$.
- **m is odd, n is odd.** Then $S_n - S_m = -x_{m+1} + x_{m+2} - \cdots + x_n = (-x_{m+1} + x_{m+2}) + \cdots + (-x_{n-1} + x_n) \leq 0$ since (x_n) is decreasing. However, $S_n - S_m = -x_{m+1} + (x_{m+2} - x_{m+3}) + \cdots + (x_{n-2} - x_{n-1}) + x_n \geq -x_{m+1} + x_n \geq -x_{m+1} > -\epsilon$, so $|S_n - S_m| < \epsilon$.
- **m is odd, n is even.** Then $S_n - S_m = -x_{m+1} + x_{m+2} - \cdots - x_n = (-x_{m+1} + x_{m+2}) + \cdots + (-x_{n-2} + x_{n+1}) - x_n \leq -x_n \leq 0$ since (x_n) is decreasing and nonnegative. But $S_n - S_m = -x_{m+1} + (x_{m+2} - x_{m+3}) + \cdots + (x_{n-1} - x_n) \geq -x_{m+1} > -\epsilon$. Thus $|S_n - S_m| < \epsilon$.

Thus, (x_n) meets the Cauchy criterion, and by the result of Problem 9, S_n converges. □

12. Let $f : \mathbb{N} \mapsto \mathbb{N}$ be bijective. Let (x_k) be a sequence in $\mathbb{R}$ and define $y_k := x_{f(k)}$. If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then

$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} y_k.$$

Proof. Denote $S_n := \sum\limits_{k=1}^{n} x_k$ and $T_n := \sum\limits_{k=1}^{n} y_k$. By presentation problem 10, since $\sum\limits_{k\geq 1} x_k$ is absolutely convergent, we know it converges to some

$x < \infty$, such that with an $\varepsilon > 0$ that is arbitrary and fixed

$$(6) \quad \exists N_1 \in \mathbb{N}, \text{ such that } \forall n \geq N_1, \ |S_n - x| < \frac{\varepsilon}{2}$$

And by presentation problem 9 along with absolute convergence, we also know that

$$(7) \quad \exists N_2 \in \mathbb{N}, \text{ such that } \forall n > m \geq N_2, \ \left| \sum_{k=m+1}^{n} |x_k| \right| = \sum_{k=m+1}^{n} |x_k| < \frac{\varepsilon}{2}$$

Let $N = \max\{N_1, N_2\}$ and let $N_3 = max\{f(k) \mid k \in [N]\}$, where $[N] = \{1, 2, 3, ..., N\}$ and f^{-1} is well defined because f is a bijection. Notice that $N_3 \geq N$ since $[N]$ contains the smallest N integers of the naturals and $f : \mathbb{N} \mapsto \mathbb{N}$. By construction of N_3, the first N_3 elements of (y_n) must contain the first N elements of (x_n). So T_{N_3} is equal to S_N plus the sum of extra terms, denoted E. There are exactly $N_3 - N$ extra terms each taking the form x_{N+i} for positive i (since we already accounted for the first N terms of (x_n) with S_n). Thus, $E \leq \sum_{k=N+1}^{M} |x_k|$ where $M = \max\{f^{-1}(k) \mid k \in [N_3]\}$. (Once again, note that $M \geq N_3 \geq N$ f is a bijection. We can use (2) to say $E < \frac{\varepsilon}{2}$, so since $T_{N_3} - S_N = E$, we know $T_{N_3} - S_N < \frac{\varepsilon}{2}$. See

$$\begin{aligned} |T_{N_3} - x| &= |T_{N_3} - S_N + S_N - x| \\ &\leq |T_{N_3} - S_N| + |S_N - x| \quad \text{by triangle inequality} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Since ε was arbitrary, any $n > N_3$ guarantees $|T_n - x| < \epsilon$, and thus, T_n converges to x. Hence, $\sum_{k \geq 1} x_k = \sum_{k \geq 1} y_k$. $\square$