Presentation Problems 1

21-355 A

Instructions: Your group should prepare a presentation for the problem corresponding to your group number. After presenting the solution and getting feedback from the class, you have until the beginning of the following class to send a Tex file with a polished form of the solution to the instructor. Make sure all group members' names are in the Tex file. There are 15 points available: 10 for the presentation and 5 for the written proof.

1. Let (a_n) be a convergent sequence. Then (a_n) is bounded. In addition, let (a_{n_k}) be a subsequence of (a_n). Then the subsequence (a_{n_k}) converges to $\lim a_n$.

2. Let (a_n) and (b_n) be sequences such that $\lim a_n = a$ and $\lim b_n = b$. Then for any $\alpha, \beta \in \mathbb{R}$, $\lim(\alpha a_n + \beta b_n) = \alpha a + \beta b$ and $\lim(a_n b_n) = ab$. Further, $\lim \frac{a_n}{b_n} = \frac{a}{b}$ provided $b \neq 0$.

3. Let (a_n) and (b_n) be sequences such that $\lim a_n = a$ and $\lim b_n = b$.

   (a) If $a_n \geq \alpha$ for all $n \in \mathbb{N}$, then $a \geq \alpha$. Similarly, if $a_n \leq \beta$ for all $n \in \mathbb{N}$, then $a \leq \beta$.

   (b) If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.

4. Let (a_n), (b_n), and (c_n) be sequences in $\mathbb{R}$ such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $\lim a_n = \lim c_n = \gamma$, then $\lim b_n = \gamma$. **Note:** It is NOT given that (b_n) converges.

5. Prove that an increasing, bounded sequence (a_n) converges to $\sup\{a_n\}$ and a decreasing, bounded sequence (b_n) converges to $\inf\{b_n\}$. Show that for any bounded sequence (a_n), the sequences (y_n) and (z_n) where

   $$y_n := \sup\{a_k : k \geq n\}$$

   and

   $$z_n := \inf\{a_k : k \geq n\}$$

   converge. (These limits are defined as the limit superior, $\limsup a_n$, and the limit inferior, $\liminf a_n$, respectively. Thus, any bounded sequence has a limit superior and limit inferior.)

6. Prove that for any bounded sequence (a_n), $\liminf a_n \leq \limsup a_n$ and show that $\liminf a_n = \limsup a_n$ if and only if $\lim a_n$ exists.

7. A Cauchy sequence is bounded and a convergent sequence is Cauchy.

8. Let $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$. Then for any $\alpha, \beta \in \mathbb{R}$,

$$\sum_{k=1}^{\infty} \alpha a_k + \beta b_k = \alpha A + \beta B.$$

9. The series $\sum_{k=1}^{\infty} x_k$ converges if and only if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $n > m \geq N$,

$$\left| \sum_{k=m+1}^{n} x_k \right| < \varepsilon.$$

   Conclude that if $\sum_{k=1}^{\infty} x_k$ converges, then $\lim_{k\to\infty} x_k = 0$.

10. If $\sum_{k=1}^{\infty} |x_k|$ converges, then $\sum_{k=1}^{\infty} x_k$ converges.

11. Let (x_n) be a decreasing sequence such that $\lim x_n = 0$. Then $\sum_{k=1}^{\infty} (-1)^{k+1} x_k$ converges.

12. Let $f : \mathbb{N} \mapsto \mathbb{N}$ be bijective. Let (x_k) be a sequence in $\mathbb{R}$ and define $y_k := x_{f(k)}$. If $\sum_{k=1}^{\infty} x_k$ converges absolutely, then

$$\sum_{k=1}^{\infty} x_k = \sum_{k=1}^{\infty} y_k.$$