SETS AND FUNCTIONS

JOSHUA BALLEW

1. Sets

As a review, we begin by considering a naive look at set theory. For our purposes, we define a *set* as a collection of objects. Except for certain sets like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, etc., we generally use capital letters $A, B, \ldots, Z$ to identify sets. We can define sets in several ways. One way is to simply list the items in the set, called the set's *elements*, for example,

$$A = \{1, 2, 7\}.$$

This method works well for sets that have finite cardinality (more on that in a bit), but what about sets that have an infinite number of items in them? We cannot very well write all of them explicitly. If there is a clear pattern, though, we may be able to "list" them, for example,

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

Moreover, we can define a set without actually listing its elements as below.

$$B = \{x \in \mathbb{C} : 6x^5 - 27x^2 + 433x + \pi = 0\}$$

Note that the colon in the definition of B means "such that" (sometimes a $|$ is used instead). Thus, B is the set of all the complex zeros of the polynomial $6x^5 - 27x^2 + 433x + \pi$. We say that $x \in E$ if x is an element of E and $x \notin E$ if x is not an element of E. For example, $2 \in A$, but $0 \notin \mathbb{N}$.

We have the following definitions of subsets and set equality.

Definition 1. A is a **subset** of B, denoted $A \subseteq B$ if and only if for all $x \in A$, $x \in B$. A is called a **proper subset** of B, denoted $A \subset B$, if and only if $A \subseteq B$ and there is some $y \in B$ such that $y \notin A$.

Definition 2. Sets A and B are equal, denoted $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Remark 3. Some texts use $A \subset B$ if A is a subset of B or equal to B and $A \subsetneq B$ if A is a proper subset of B.

Definition 2 provides a method for proving that two sets are equal: prove that each is a subset of the other. Next, we define two ways to combine sets: union and intersection.

Definition 4. The **union** of sets A and B is the set of all elements that are in A or in B. It is denoted as

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$

Definition 5. The **intersection** of sets A and B is the set of all elements that are in A and in B. It is denoted as

$$A \cap B := \{x : x \in A \text{ and } x \in B\}.$$

Remark 6. Some texts use the notation $A \sqcup B$ for the **disjoint union** of A and B, that is,

$$A \sqcup B := \{x : x \in A \cup B \text{ and } x \notin A \cap B\}.$$

Definition 7. Let A and B be sets. We define the set of ordered pairs, called the **Cartesian product** of A and B, denoted $A \times B$ as

$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}.$$

We can also consider unions and complements of large collections of sets. Let I be an indexing set, such that we have a collection of sets A_i for each $i \in I$. (Note that I can be finite or infinite.) Then we have the following definition.

Definition 8. The union of a collection of sets is

$$\bigcup_{i \in I} A_i := \{x : \exists i \in I \text{ such that } x \in A_i\}.$$

The intersection of a collection of sets is

$$\bigcap_{i \in I} A_i := \{x : x \in A_i \ \forall i \in I\}.$$

Exercise 1. Prove that for any set A, we have that $A \subseteq A$ and $\emptyset \subseteq A$.

Exercise 2. Is $\emptyset \in A$ for an arbitrary set A?

Exercise 3. Does $\{\emptyset\} = \emptyset$?

Exercise 4. Prove the following.

(1) $A \cap B = B \cap A$
(2) $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I}(A \cap B_i)$
(3) $A \cup \bigcap_{i \in I} C_i = \bigcap_{i \in I}(A \cup C_i)$

Exercise 5. For each $x \in \mathbb{R}$, let $A_x := [x-1, x+1]$. Find the following intersections and unions.

(1) $\bigcap_{x \in \mathbb{R}} A_x$
(2) $\bigcap_{x \in [0,1)} A_x$
(3) $\bigcup_{x \in \mathbb{R}} A_x$
(4) $\bigcup_{x \in [0,1)} A_x$

Next, we consider the possibility that we have some set A such that $A \in A$. For our purposes, we will call such a set "abnormal" and any set such that $A \notin A$ a "normal" set. Consider the set $X := \{A : A \text{ is a normal set}\}$. Show that

Exercise 6. If X is normal, then $X \in X$.

Exercise 7. If X is abnormal, then $X \notin X$.

Exercise 8. Discuss whether X normal or abnormal.

The field of set theory devotes much effort on the problem of defining exactly what a set is. There is a set of axioms called the Zermelo-Frankel (ZF) axioms which most mathematicians use to define sets. In the ZF system, it is not possible to have $A \in A$. To avoid this and other logical problems, we often define a universe $\mathcal{U}$ in which we work. We can then define the complement of a set

Definition 9. Let $A \subseteq \mathcal{U}$ for a given universe $\mathcal{U}$. The **complement** of A is the set

$$A^c := \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

If $\mathcal{U}$ is understood but not explicitly mentioned, we often write

$$A^c := \{x : x \notin A\}.$$

Now that the complement of a set is defined, we can define the set theoretic difference.

Definition 10. The **set theoretic difference** $A - B$ is defined as

$$A - B := A \cap B^c.$$

We also define the power set of A as follows.

Definition 11. The power set of A, denoted as $\mathcal{P}(A)$ is defined as

$$\mathcal{P}(A) := \{B : B \subseteq A\}.$$

Exercise 9. Prove the following

(1) $\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$.
(2) $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$.
(3) $(A^c)^c = A$.

Exercise 10. Prove or disprove:

(1) $(A - B) - C = A - (B \cap C)$.
(2) $(A - B) - C = A - (B \cup C)$.

Exercise 11. How many elements does $\mathcal{P}(A)$ have if

$$A = \{1, 2, 3, \ldots, n\}?$$

2. Functions

Now that we have defined sets, we define functions between them

Definition 12. f is a **function** from set X to set Y, taking elements $x \in X$ to elements $y \in Y$ if and only if for each $x \in X$, there exists exactly one $y \in Y$ such that f maps x to y. We denote this as $f(x) = y$ and write $f : X \mapsto Y$. X is called the **domain** of f and Y is called the **codomain**. The set of $y \in Y$ such that there is some $x \in X$ such that $f(x) = y$ is called the **range** of f.

We also have special types of functions: one to one and onto.

Definition 13. We say f is **one to one** or **injective** if $a = b$ whenever $f(a) = f(b)$. We say f is **onto** or **surjective** if its range equals its codomain. f is called **bijective** if and only if f is one to one and onto.

Taking subsets $A \subseteq X$ and $B \subseteq Y$ of the domain and codomain, respectively, we can define images and preimages of sets.

Definition 14. Let $A \subseteq X$ and $B \subseteq Y$, with $f : X \mapsto Y$. We define the **image** of A, denoted $f(A)$ as

$$f(A) := \{y \in Y : y = f(x) \text{ for some } x \in A\}$$

and the **preimage** of B, denoted $f^{-1}(B)$ as

$$f^{-1}(B) := \{x \in X : f(x) = b \text{ for some } b \in B\}.$$

Suppose that we have a function $f : X \mapsto Y$ and a function $g : Y \mapsto Z$ for sets X, Y, and Z. We can define a new function $h : X \mapsto Z$ by taking the composition of f and g.

Definition 15. Let $f : X \mapsto Y$ and $g : Y \mapsto Z$. Then the **composition** of g and f, denoted $g \circ f$ maps X to Z and is defined as

$$(g \circ f)(x) = g(f(x)).$$

Exercise 12. Let $f : \mathbb{R} \mapsto \mathbb{R}$ be given by $f(x) = x^2$. Calculate

(1) $f((0, 3])$
(2) $f^{-1}((0, 3])$
(3) $f^{-1}((-1, 2))$

Exercise 13. Let $f : X \mapsto Y$ and $A, B \subseteq X$. Prove or disprove:

(1) $f(A \cup B) = f(A) \cup f(B)$
(2) $f(A \cap B) \subseteq f(A) \cap f(B)$
(3) $f(A \cap B) = f(A) \cap f(B)$.

Exercise 14. Let $f : X \mapsto Y$ and $C, D \subseteq Y$. Prove or disprove the following.

(1) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
(2) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

Exercise 15. Let $f : X \mapsto Y$. Prove that f is one to one if and only if $f^{-1}(f(A)) = A$ for all $A \subseteq X$.

Exercise 16. Let $f : X \mapsto Y$. Prove that f is onto if and only if $f(f^{-1}(B)) = B$ for all $B \subseteq Y$.

Exercise 17. Let $f : X \mapsto Y$ be one to one and onto. Show there is a function $f^{-1} : Y \mapsto X$ such that $f(f^{-1}(y)) = y$ for all $y \in Y$ and $f^{-1}(f(x)) = x$ for all $x \in X$. Prove that f^{-1} is itself one to one and onto. f^{-1} is called the **inverse** of f.

3. Cardinality

The next idea about sets we discuss is their size, specifically, how many elements they have. For sets that have a finite number of elements, which we call finite sets, this is a simple thing: we just count the elements. However, for sets with an infinite number of elements (infinite sets), the idea of the number of elements is more subtle. To handle this, we introduce the idea of cardinality. To motivate the definition, we consider finite sets $A := \{a, b, c, d, e\}$ and $B := \{1, 2, 3, 4, 5\}$. Clearly, A and B each have five elements. But note that we can form the bijection (verify) $f : A \mapsto B$ where $f(a) = 1$, $f(b) = 2$, and so on. This observation motivates the following definition of cardinality.

Definition 16. We say that sets A with cardinality $\widehat{A}$ and B with cardinality $\widehat{B}$ have the same **cardinality**, denoted $\widehat{A} = \widehat{B}$ if and only if there exists a bijection $f : A \mapsto B$. We also say A is **equinumerous** to B.

With this definition of cardinality, we can define a finite set with a bit more rigor. We say a set A is **finite** if and only if it is the empty set (which we say has cardinality zero, or $\widehat{\emptyset} = 0$) or it is equinumerous to a set $\{1, 2, 3, \ldots, n\}$ for some $n \in \mathbb{N}$. In this second case we say $\widehat{A} = n$. If A is not finite, it is **infinite**. However, we divide sets of infinite cardinality into two camps: countably infinite or uncountably infinite/uncountable. (Note that a countable set can be finite or countably infinite.) A set is **countably infinite** if and only if it is equinumerous to $\mathbb{N}$. Otherwise, an infinite set is **uncountable**. To close this part, we prove a proposition using Cantor's diagonalization argument.

Proposition 17. *Let A and B be countable sets. Then $A \times B$ is countable.*

Proof. If A or B is empty, then $A \times B$ is empty (verify), and we are done. Otherwise, we can write

$$A = \{a_1, a_2, \ldots\}$$
$$B = \{b_1, b_2, \ldots\},$$

where these lists may or may not terminate (see the exercises). Because of this, we can write the ordered pairs of $A \times B$ in an array as

$$\left\{ \begin{array}{cccc} (a_1, b_1) & (a_1, b_2) & (a_1, b_3) & \cdots \\ (a_2, b_1) & (a_2, b_2) & (a_2, b_3) & \cdots \\ (a_3, b_1) & (a_3, b_2) & (a_3, b_3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array} \right\}.$$

We can now list the elements by going along diagonals going up to the right. We start with (a_1, b_1) and continue with the next diagonal going up to the right with (a_2, b_1) and (a_1, b_2) and we continue. We will then get a list of elements of $A \times B$

$$\{(a_1, b_1), (a_2, b_1), (a_1, b_2), (a_3, b_1), (a_2, b_2), (a_1, b_3) \ldots\}.$$

Since it can be written out as a list, $A \times B$ is countable. □

Prove the following.

Exercise 18. Let A be a set. Then A is countably infinite if and only if it can be written as

$$A = \{a_1, a_2, \ldots\}.$$

Conclude that a set A is countable if and only if $A = \emptyset$ or $A = \{a_1, a_2, \ldots\}$, which is a list that may or may not terminate.

Exercise 19. If $A \subseteq B$ and B is countable, then A is countable.

Exercise 20. If $A \subseteq B$ and A is uncountable, then B is uncountable.

Exercise 21. $\mathbb{Q}$ is countably infinite.

Exercise 22. If I is countable and for all $i \in I$, A_i is countable, then $\bigcup_{i \in I} A_i$ is countable. **Hint:** Use the Cantor diagonalization argument.

Exercise 23. $[0, 1]$ is uncountable. Conclude $\mathbb{R}$ is uncountable.

Exercise 24. $\mathbb{R} - \mathbb{Q}$ is uncountable.

3.1. **Comparing Cardinality.** So far, we have considered how to show two (or more) sets are equinumerous by using the definition, namely, finding a bijection between the sets. However, constructing a bijection or showing one exists can be difficult. But there is an easier way. First, we define a less than or equal to operator on set cardinalities.

Definition 18. Given two sets A and B, we say $\widehat{A} \leq \widehat{B}$ if and only if there is an one to one function $f : A \mapsto B$. If $\widehat{A} \leq \widehat{B}$ but $\widehat{A} \neq \widehat{B}$, we say $\widehat{A} < \widehat{B}$.

Since we know that for two real numbers x and y, $x \leq y$ and $y \leq x$ if and only if $x = y$, you might be wondering if this holds for cardinality. Proving that equality implies the two less than or equal to statements is easy, and is left as an exercise. The other implication, however, is significantly harder to prove.

Theorem 19 (Cantor, Bernstein, Schroeder)**.** *Let A and B be sets such that $\widehat{A} \leq \widehat{B}$ and $\widehat{B} \leq \widehat{A}$. Then $\widehat{A} = \widehat{B}$.*

Proof. Let $\widehat{A} \leq \widehat{B}$ and $\widehat{B} \leq \widehat{A}$. Then there are one to one mappings $f : A \mapsto B$ and $g : B \mapsto A$. Our goal is to produce a bijective mapping $h : A \mapsto B$. Take $b \in B$. If there is some $a \in A$ such that $f(a) = b$, we will call a the first ancestor of b. Likewise, if there is also some $c \in B$ such that $g(c) = a$, then c is a first ancestor to a and a second ancestor to b, and so on. We note each ancestor is unique since f and g are one to one.

We partition A, leaving it to the reader to verify that the following is a partition of A, (recall that a collection of sets $\{A_i\}_{i \in I}$ is a partition of A if and only if $\bigcup_{i \in I} A_i = A$ and for all $i, j \in I$, $A_i \cap A_j = \emptyset$ if $i \neq j$):

$$A = A_o \cup A_e \cup A_\infty$$

where

$$\begin{aligned} A_o &:= \{a \in A : a \text{ has an odd number of ancestors}\} \\ A_e &:= \{a \in A : a \text{ has an even number of ancestors}\} \\ A_\infty &:= \{a \in A : a \text{ has an infinite number of ancestors}\}. \end{aligned}$$

We similarly partition B as $B = B_o \cup B_e \cup B_\infty$.

Next, we show that f maps A_e to B_o and A_∞ to B_∞ and both of these restrictions are onto (we know already that f is one to one on both of these restrictions since it is one to one on A). Let $b \in B_o$. Then b has an odd number of ancestors; in particular, it has a first ancestor which we will call a, in A, such that $f(a) = b$. Since b has an odd number of ancestors, a must have as ancestors all of b's ancestors except itself. Thus, a has one fewer ancestor than b, which means a has an even number of ancestors. Thus $a \in A_e$, and therefore f restricted to A_e is a bijective map to B_o.

To prove f is an onto mapping from A_∞ to B_∞, we take some arbitrary $d \in B_\infty$. Since it has an infinite number of ancestors, it has a first ancestor $\gamma \in A$ and $f(\gamma) = d$. However, γ will only have one fewer ancestor than d does, therefore it has an infinite number of ancestors as well and is in A_∞. Thus, f maps $A_e \cup A_\infty$ to $B_o \cup B_\infty$ bijectively.

So the only part of the mapping h we need to define is from A_o to B_e. We cannot use f as some elements of B_e may have zero ancestors. Instead, we will show that g maps B_e bijectively to A_o, and use g's inverse. We already know that g is one to one, so we only need to show that each element a of A_o has some $b \in B_e$ that maps to it. But if $a \in A_o$, it has an odd number of ancestors, so it has at least one ancestor, $b \in B$ where $g(b) = a$. However, b has one fewer ancestor than a does, so b has an even number of ancestors. Thus, g is a bijective mapping from B_e to A_o and therefore g^{-1} is bijective from A_o to B_e.

In conclusion we have the bijective mapping $h : A \mapsto B$ where

$$h(a) := \begin{cases} f(a), & a \in A_e \cup A_\infty \\ (g|_{B_e})^{-1}(a), & a \in A_o \end{cases}.$$

Note that we can only consider the inverse of g restricted to B_e as this is where we have shown g is bijective, and thus has an inverse. ☐

Prove the following, assuming A, B, and C are arbitrary sets:

Exercise 25. If $\widehat{A} = \widehat{B}$, then $\widehat{A} \leq \widehat{B}$.

Exercise 26. $\widehat{A} = \widehat{A}$.

Exercise 27. If $\widehat{A} \leq \widehat{B}$ and $\widehat{B} \leq \widehat{C}$, then $\widehat{A} \leq \widehat{C}$.

Exercise 28. $\widehat{A} < \widehat{\mathcal{P}(A)}$. **Hint:** You will need to show that there is no onto mapping from A to $\mathcal{P}(A)$. Prove this by contradiction by assuming the existence of an onto $g : A \mapsto \mathcal{P}(A)$ and let $C := \{a \in A : a \notin g(a)\} \in \mathcal{P}(A)$. Taking $c \in A$ such that $g(c) = C$ (why does this c exist?), consider if $c \in C$ or $c \notin C$.

Exercise 29. $\widehat{\mathbb{R}} = \widehat{\mathbb{R}^2}$.

Exercise 30. $\widehat{[0,1]} = \widehat{(0,1)} = \widehat{\mathbb{R}}$. Note that $[0,1]$ and $(0,1)$ can be replaced by arbitrary infinite or finite intervals.

Exercise 31. Let $\mathbb{P} := \{p(x) : p(x) \text{ is a polynomial with integer coefficients}\}$. Then $\widehat{\mathbb{P}} = \widehat{\mathbb{N}}$.