21-241 Lec5 **Exam 2** **October 21, 2015**

> Answer the following questions on the given sheets of paper. Make sure that each sheet corresponds to only ONE problem. You may use the backs of the sheets of paper if you wish. You must show your work to receive full credit. Make sure your answers are legible and clear. Calculators that cannot access a computer network are allowed. No notes, crib sheets, books, phones, computers, or other electronic devices are allowed. You have 50 minutes for the exam. Staple (in the upper left corner) this sheet to the top of your work when you turn in the exam, and then place your exam on the pile for your section.

Name: ______________________________

Section: ______________________________

1. (a) (20 points) Let S be a subspace of $\mathbb{R}^n$ and $T : \mathbb{R}^n \mapsto \mathbb{R}^m$ be a linear transformation. Prove that
$$T(S) := \{\mathbf{y} \in \mathbb{R}^m | \exists \mathbf{x} \in S \text{ such that } T(\mathbf{x}) = \mathbf{y}\}$$
is a subspace of $\mathbb{R}^m$.

   (b) (5 points) Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear transformation with standard matrix representation A, that is, $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$. Show that if A is invertible, then $T(\mathbb{R}^n) = \mathbb{R}^n$.

2. Consider the matrix
$$A = \begin{bmatrix} -2 & -2 & 3 \\ 0 & 3 & -2 \\ 0 & -1 & 2 \end{bmatrix}.$$
   (a) (10 points) Find the characteristic polynomial for A.
   (b) (5 points) Determine the eigenvalues of A.
   (c) (10 points) Find a basis for the eigenspace E_1.
   (d) (5 points) Explain why $\text{rank}(A) = 3$.

3. Let l be the line in $\mathbb{R}^2$ though the origin with directional vector $\mathbf{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Let $P_l : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the projection of a vector onto line l. That is, $P_l(\mathbf{v}) = \text{proj}_{\mathbf{d}}(\mathbf{v})$.
   (a) (10 points) Determine the standard matrix B for P_l.
   (b) (10 points) Determine a basis for $\text{null}(B)$.
   (c) (5 points) Calculate $\text{nullity}(B)$ and explain why or why not P_l is invertible.

4. Let A be a 2×2 matrix with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ corresponding to eigenvalues $\lambda_1 = 4$ and $\lambda_2 = -1$, respectively. Let $\mathbf{x} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$.
   (a) (5 points) Write $\mathbf{x}$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
   (b) (10 points) What is $A^k\mathbf{x}$ for $k \in \mathbb{N}$?
   (c) (5 points) Explain why A can have only the two given eigenvalues and why the formula in your answer for part (b) will work for *any* integer k.