Answer the following questions on the given sheets of paper. Make sure that each sheet corresponds to only ONE problem. You may use the backs of the sheets of paper if you wish. You must show your work to receive full credit. Make sure your answers are legible and clear. Calculators that cannot access a computer network are allowed. No notes, crib sheets, books, phones, computers, or other electronic devices are allowed. You have 3 hours for the exam. Staple (in the upper left corner) this sheet to the top of your work when you turn in the exam, and then place your exam on the pile for your section.
Unless otherwise stated, assume all vector spaces are real vector spaces.

Name: ______________________________

Section: ______________________________

1. Let A be an $n \times n$ matrix, and $\mathbf{b} \in \mathbb{R}^n$, $\mathbf{b} \neq \mathbf{0}$.
   (a) (10 points) Show that if $\mathbf{x}_p$ is a solution to $A\mathbf{x} = \mathbf{b}$ and $\mathbf{x}_h$ solves the corresponding homogeneous problem $A\mathbf{x} = \mathbf{0}$, then $\mathbf{x}_p + \mathbf{x}_h$ solves $A\mathbf{x} = \mathbf{b}$.
   (b) (10 points) Use part (a) to show that if A is non-invertible and $\mathbf{x}_p$ solves $A\mathbf{x} = \mathbf{b}$, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions. **Hint:** What do you know about null(A)?
   (c) (5 points) Give an example of a 2×3 matrix with all rows having a leading one that is in row echelon form but is NOT in reduced row echelon form.

2. (a) (15 points) Find all real values s such that
$$\begin{bmatrix} s & 1 & 0 \\ 1 & s & 1 \\ 0 & 1 & s \end{bmatrix}$$
   is singular.
   (b) (5 points) Prove or disprove that
$$\begin{bmatrix} 4 & 1 \\ 0 & -3 \end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}.$$
   (c) (5 points) Let B be $n \times n$. Prove that the entries on the diagonal of $B - B^T$ are all zeros.

3. Let
$$B = \begin{bmatrix} 1 & -2 & 0 & 3 & -4 \\ 3 & 2 & 8 & 1 & 4 \\ 2 & 3 & 7 & 2 & 3 \\ -1 & 2 & 0 & 4 & -3 \end{bmatrix}.$$
   (a) (10 points) Find a basis for the column space of B.
   (b) (8 points) Find a basis for the row space of B.
   (c) (7 points) Calculate nullity(B). Explain how you obtained your answer.

4. (a) (10 points) Let $V = \mathbb{R}^2$ with the usual addition and scalar multiplication given by

$$c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ cy \end{bmatrix}$$

for any $c \in \mathbb{R}$. Prove or disprove that V with the operations above is a real vector space.

(b) (15 points) Let

$$W = \left\{ \begin{bmatrix} a \\ a+b \\ b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}.$$

Use the Gram-Schmidt process to find an orthogonal basis of W. You may assume without proof that W is a subspace of $\mathbb{R}^3$. **Hint:** Start with finding a basis of W.

5. Let

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

(a) (15 points) Find an orthogonal matrix Q and diagonal matrix D such that $D = Q^T C Q$.

(b) (10 points) Let

$$\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}.$$

Find $[\mathbf{v}]_B$ where B is a basis of $\mathbb{R}^3$ formed by the columns of Q. Be sure to indicate clearly what basis you are using.

6. (a) (10 points) Let $V \cong W$ and let V_1 be a subspace of V. Show that for any isomorphism $L : V \mapsto W$, $V_1 \cong L(V_1)$.

(b) Let $L : \mathbb{R}^2 \mapsto \mathbb{R}^3$ be defined by

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ x+y \\ y \end{bmatrix}.$$

i. (5 points) Give the matrix of L with respect to the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$.

ii. (5 points) Find $\ker(L)$.

iii. (5 points) Determine if L is injective and if L is surjective.

7. (a) (8 points) Let V be an inner product space with inner product $\langle \cdot, \cdot \rangle$ with associated norm $\| \cdot \|$. Prove that for $\mathbf{u}, \mathbf{v} \in V$,

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.$$

(b) (5 points) Prove that if $A = A^{-1}$, then $\det(A) = \pm 1$.

(c) (12 points) Let Q be an orthogonal $n \times n$ real matrix. Calculate (showing your work and explaining your reasoning), $\|Q\|_2$.

8. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -2 & 1 \end{bmatrix}.$$

(a) (10 points) Find the singular values of A.

(b) (5 points) Write the quadratic form associated with the matrix AA^T (not $A^T A$).

(c) (10 points) Find an equivalent quadratic form to your answer in (b) that has no cross terms.