21-241 Lec5 **Exam 4** **December 4, 2015**

Answer the following questions on the given sheets of paper. Make sure that each sheet corresponds to only ONE problem. You may use the backs of the sheets of paper if you wish. You must show your work to receive full credit. Make sure your answers are legible and clear. Calculators that cannot access a computer network are allowed. No notes, crib sheets, books, phones, computers, or other electronic devices are allowed. You have 50 minutes for the exam. Staple (in the upper left corner) this sheet to the top of your work when you turn in the exam, and then place your exam on the pile for your section. **Unless otherwise stated, assume all vector spaces are real vector spaces.**

Name: ________________________________

Section: ________________________________

1. Let $B = \{x - x^2, x^2 - 1, 1 + x\}$ and $C = \{x^2, x - 1, x + 1\}$.
   (a) (10 points) Prove that B is a basis of P_2.
   (b) (10 points) Calculate $P_{B \leftarrow C}$. You may assume without proof that C is a basis of P_2.
   (c) (5 points) Use $P_{B \leftarrow C}$ above to find $[x^2 + x + 1]_B$. (You must use the change of basis matrix to receive credit.)
2. Let $L : P_2 \mapsto P_1$ be defined by $L(p(x)) = p'(x)$ for $p(x) \in P_2$.
   (a) (12 points) Let $B = \{1, x, x^2\}$ be a basis for P_2 and $C = \{x, x+1\}$ be a basis for P_1. Find $[L]_{C \leftarrow B}$.
   (b) (8 points) Use the matrix for L with respect to bases B and C to find $[(4x^2 - 2x + 1)']_C$. (You must use the matrix for L with respect to B and C to receive credit.)
   (c) (5 points) Calculate nullity L.
3. (a) (15 points) Let V be an n-dimensional vector space with basis B. Show that $L : V \mapsto \mathbb{R}^n$ where $L(\mathbf{x}) = [\mathbf{x}]_B$ is an isomorphism.
   (b) (10 points) Let $\mathbf{u} \in \mathbb{R}^2$, $\mathbf{u} \neq \mathbf{0}$. Show that for $L : \mathbb{R}^2 \mapsto \mathbb{R}^2$ where $L(\mathbf{v}) = \text{proj}_{\mathbf{u}}(\mathbf{v})$, that $L(\mathbb{R}^2) \cong \mathbb{R}$.
4. (a) (5 points) Let V be a real inner product space with unit vectors $\mathbf{u}$ and $\mathbf{v}$. Prove that $\langle \mathbf{u}, \mathbf{v} \rangle \geq -1$.
   (b) (10 points) Consider the real inner product space $C[0, 1]$ with the usual addition and scalar multiplication with the inner product
   $$\langle f, g \rangle := \int_0^1 f(x)g(x)\, dx.$$
   (You may assume without proof this is an inner product.) Find conditions on constants a and b such that $p(x) = \sqrt{x}$ is orthogonal to $q(x) = a + bx$.
   (c) (10 points) For the inner product space in part (b), find the distance between f and g where $f(x) = x$ and $g(x) = x^2$.