21-241 Lec5 **Exam 3** **November 16, 2015**

Answer the following questions on the given sheets of paper. Make sure that each sheet corresponds to only ONE problem. You may use the backs of the sheets of paper if you wish. You must show your work to receive full credit. Make sure your answers are legible and clear. Calculators that cannot access a computer network are allowed. No notes, crib sheets, books, phones, computers, or other electronic devices are allowed. You have 50 minutes for the exam. Staple (in the upper left corner) this sheet to the top of your work when you turn in the exam, and then place your exam on the pile for your section.

Name: ____________________

Section: ____________________

1. Let
$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\}.$$
   (a) (7 points) Prove that B is an orthogonal basis of $\mathbb{R}^3$.

   (b) (6 points) Let
   $$\mathbf{v} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}.$$
   Find $\left[\mathbf{v}\right]_B$.

   (c) (5 points) Let A and B be real, orthogonal $n \times n$ matrices. Prove that $A(A^T + B^T)B = A + B$.

   (d) (7 points) Use the result of part (c) to show that for real, orthogonal $n \times n$ matrices A and B, if $\det A + \det B = 0$, then $A + B$ is singular.

2. Let A be a real 3×3 matrix with eigenvalues of -1 and 2. The eigenspaces are
$$E_{-1} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 5 \end{bmatrix} \right\}$$
   and
$$E_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$
   (a) (10 points) Find a diagonal matrix D and invertible matrix P such that $D = P^{-1}AP$. Make sure to prove that P is invertible.

   (b) (5 points) Explain why there are no orthogonal matrix Q and diagonal matrix D such that $D = Q^{-1}AQ$.

   (c) (10 points) Let $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Calculate $\text{proj}_{E_{-1}}(\mathbf{v})$ and $\text{perp}_{E_{-1}}(\mathbf{v})$.

3. (a) (10 points) Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$. Calculate $\text{perp}_{\mathbf{u}}(\mathbf{v})$.

   (b) (15 points) Use the result of part (a) to find a matrix Q with orthonormal columns and upper triangular matrix R such that $A = QR$ where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 2 \end{bmatrix}.$$

4. (a) (10 points) Let $V = \mathbb{R}^2$ with scalar multiplication defined in the usual way and addition defined such that

$$\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a + c \\ 0 \end{bmatrix}.$$

   Prove or disprove that V with the operations described above is a real vector space.

   (b) (10 points) Let $\mathcal{F}$ be the real vector space of functions $f : (0, \infty) \mapsto \mathbb{R}$ with the usual addition and scalar multiplication. Prove or disprove that $A = \left\{1, \ln(2x), \ln(x^2)\right\}$ is a linearly independent subset of $\mathcal{F}$.

   (c) (5 points) Let $\mathbf{0} \neq \mathbf{u} \in V$ where V is a vector space over some scalar field. Let c and d be scalars. Show that if $c\mathbf{u} = d\mathbf{u}$, then $c = d$.