# Partial Fraction Decomposition and Integration

Joshua Ballew

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A rational function $r(x)$ is a function that can be written as $\frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials without any common factors. It will be assumed throughout this document that the degree of $p(x)$ is less than the degree of $q(x)$. (If this is not the case, polynomial long division will give a sum of a polynomial and a new rational function a lower degree in its numerator than in its denominator.) Also, only rational functions with denominators $q(x)$ that can be written as a product of linear and quadratic terms, some of which may be repeated, are considered. The partial fraction decomposition expresses a rational function as the sum of rational functions with simpler denominators. These simpler denominators allow us to more easily integrate the function.

There are two main steps in determining the partial fraction decomposition of a rational function. The first step is to determine the form of the partial fraction decomposition. The second is to determine the coefficients. After finding the partial fraction decomposition, we can integrate.

## 1 Determining the Partial Fraction Form

The first step in finding the partial fraction decomposition is to determine its form. This depends **only on the form of the denominator**. First, the denominator must be factored completely. Next, we look at the factors of the denominator. Any linear factor $bx - c$ that is not repeated contributes a term of the form

$$\frac{A}{bx-c}.$$

Any linear factor that is repeated, say $(bx-c)^n$ contributes terms

$$\frac{A_1}{bx-c} + \frac{A_2}{(bx-c)^2} + \cdots + \frac{A_n}{(bx-c)^n}.$$

A non-repeated quadratic factor $ax^2 + bx + c$ contributes a term of

$$\frac{Bx+C}{ax^2+bx+c}$$

and a repeated quadratic factor $(ax^2+bx+c)^n$ contributes terms

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}.$$

To find the form of the partial fraction decomposition, we add the contributions from all of the factors.

**Example 1.** Find the form of the partial fraction decomposition of

$$\frac{2x}{(x-1)(x+3)}.$$

*Solution.* We see that the denominator is already factored, saving us this step. We also notice that each denominator factor is linear and not repeated. Thus, since $(x-1)$ contributes a term of the form

$$\frac{A}{x-1}$$

and $(x+3)$ contributes

$$\frac{B}{x+3},$$

the partial fraction decomposition is of the form

$$\frac{2x}{(x-1)(x+3)}=\frac{A}{x-1}+\frac{B}{x+3}$$

with $A$ and $B$ as constants to be determined. □

**Example 2.** Find the form of the partial fraction decomposition of

$$\frac{x^4+1}{x(x^2+1)^2}.$$

*Solution.* Again, we see that the denominator is factored as far as it can go, so we now look for what each factor contributes. $x$ is a non-repeated linear factor; it contributes

$$\frac{A}{x}.$$

$(x^2+1)^2$ is a repeated quadratic factor with $n=2$. Thus, it contributes

$$\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}.$$

Therefore, the partial fraction decomposition is of the form

$$\frac{x^4+1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}$$

for some constants $A$, $B$, $C$, $D$, and $E$ to be determined. □

## 2 Determining the Coefficients of the Expansion

Now that we know how to determine the form of a partial fraction expansion, we now need to determine the coefficients. There are two main methods to consider. For both methods, the first thing we do is multiply by the denominator of the function we are trying to decompose.

**Method 1: Exapnd Everything.** In this method, after multiplying by the denominator, we expand all the terms out and then regroup by the powers of $x$: constant terms, $x$, $x^2$, and so on. We then set the coefficient for each power of $x$ on the left to the corresponding coefficient on the right. We then get a system of linear equations and solve for the coefficients $A$, $B$, $C$, and so on. This method will work for all the partial fraction problems you will see in this course.

**Method 2:** In this method, after multiplying the denominator, we know that since the equation holds for any value of $x$, we can pick specific values of $x$ to make terms zero to more easily solve for the coefficients. While there is less computation involved with this method, it tends not to work well with quadratic factors.

Now, we look at some examples.

**Example 3.** Find the partial fraction decomposition for Example 1.

*Solution.* From our previous work, we know we need to find $A$ and $B$ such that

$$\frac{2x}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}.$$

Multiplying both sides by the common denominator $(x-1)(x-3)$ we get

$$2x = A(x+3) + B(x-1).$$

**Method 1:** Expanding out, we get

$$2x = Ax + 3A + Bx - B = x(A+B) + (3A - B).$$

Equating the coefficients for the powers of $x$, we get the system of equations

$$A + B = 2$$

and

$$3A - B = 0.$$

From the second equation, we see $B = 3A$, and plugging this into the first equation gives

$$2 = A + 3A = 4A,$$

so $A = \frac{1}{2}$. This means that $B = \frac{3}{2}$ and the partial fraction decomposition is

$$\frac{2x}{(x-1)(x+3)} = \frac{1}{2(x-1)} + \frac{3}{2(x+3)}.$$

**Method 2:** Using

$$2x = A(x+3) + B(x-1),$$

we see that the $A$ term on the right goes away for $x = -3$ and the $B$ term on the right disappears for $x = 1$. Since the equation holds for all $x$, it must hold for these two values as well. For $x = -3$, we get

$$-6 = -4B,$$

so $B = \frac{3}{2}$. For $x = 1$, we get

$$2 = 4A,$$

so $A = \frac{1}{2}$. This is the same answer as we got from the first method.

□

**Example 4.** Find the partial fraction decomposition for the function in Example 2.

*Solution.* We need to find $A$, $B$, $C$, $D$, and $E$, such that

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2},$$

or, after multiplying both sides by the common denominator,

$$x^4 + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$$

We first consider the second method for finding the coefficients.
**Method 2:** We see that all but the $A$ term on the right disappears for $x = 0$. Using $x = 0$, we get

$$1 = A.$$

However, there is no value of $x$ that is going to make $x^2 + 1$ go away. Finding $A = 1$ is as far as this method will take us. So we turn to the first method.

**Method 1:** We expand out to get

$$\begin{aligned} x^4 + 1 &= Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex \\ &= x^4(A+B) + x^3(C) + x^2(2A+B+D) + x(C+E) + A. \end{aligned}$$

Equating the coefficients on left and right, we get

$$A + B = 1 \tag{1}$$
$$C = 0 \tag{2}$$
$$2A + B + D = 0 \tag{3}$$
$$C + E = 0 \tag{4}$$
$$A = 1. \tag{5}$$

From (2), we get immediately $C = 0$, and from (5), we get $A = 1$. Knowing $C$, we get from (4) that $E = 0$, and from (1), that $B = 0$ since $A = 1$. Thus, (3) gives that $D = -2$. Therefore,

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{1}{x} - \frac{2x}{(x^2+1)^2}.$$

Make sure to add the fractions on the right to check that the answer is correct.

□

# 3 Integrating

The hard work, the algebra, is finished. All that is left to do is integrate using the techniques we already know. Typically, we will use the power rule, the log rule, and the arctangent formula. We may also have to do a $u$-substitution as well.

**Example 5.** Find

$$\int \frac{2x}{(x-1)(x+3)}\ dx.$$

*Solution.* From our previous work, we know

$$\frac{2x}{(x-1)(x+3)} = \frac{1}{2(x-1)} + \frac{3}{2(x+3)}.$$

Thus,

$$\begin{aligned}\int \frac{2x}{(x-1)(x+3)}\ dx &= \int \frac{1}{2(x-1)} + \frac{3}{2(x+3)}\ dx \\ &= \frac{1}{2}\int \frac{1}{x-1}\ dx + \frac{3}{2}\int \frac{1}{x+3}\ dx = \frac{1}{2}\ln|x| + \frac{3}{2}\ln|x+3| + K.\end{aligned}$$

□

**Example 6.** Find

$$\int \frac{x^4+1}{x(x^2+1)^2}\ dx.$$

*Solution.* From our previous work, we have

$$\frac{x^4+1}{x(x^2+1)^2} = \frac{1}{x} - \frac{2x}{(x^2+1)^2}.$$

Thus,

$$\begin{aligned}\int \frac{x^4+1}{x(x^2+1)^2}\ dx &= \int \frac{1}{x} - \frac{2x}{(x^2+1)^2}\ dx \\ &= \int \frac{1}{x}\ dx - \int \frac{2x}{(x^2+1)^2}\ dx.\end{aligned}$$

Using the substitution $u = x^2 + 1$ in the second integral on the right yields

$$\ln|x| + \frac{1}{x^2+1} + K.$$

$\square$