THE MORSE COMPLEX FOR A MORSE FUNCTION ON A MANIFOLD WITH CORNERS

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ABSTRACT. A Morse function f on a manifold with corners M allows the characterization of the Morse data for a critical point by the Morse index. In fact, a modified gradient flow allows a proof of the Morse theorems in a manner similar to that of classical Morse theory. It follows that M is homotopy equivalent to a CW-complex with one cell of dimension λ for each essential critical point of index λ. The goal of this article is to determine the boundary maps of this CW-complex, in the case where M is compact.

First, the boundary maps are defined in terms of the modified gradient flow. This is complicated by the fact that globally, we have only a forward flow $\varphi : [0, \infty) \times M \to M$. In the neighborhood of an essential critical point p, however, we can define a flow which effectively reverses the flow φ in the stratum containing p.

Then a transversality condition is imposed which insures that the attaching map is non-degenerate in a neighborhood of each critical point. The degree is then interpreted as a sum of trajectories connecting two critical points each counted with a multiplicity determined by a choice of orientations on the tangent spaces of the unstable manifold at each critical point.

1. INTRODUCTION

Goresky and MacPherson's *Stratified Morse Theory* ([GM]) represented a great step forward in extending the ideas of Morse, Thom and Smale to more general topological spaces. The complexities involved in dealing with stratified spaces prompted Goresky and MacPherson comment on the nostalgia their Stratified Morse Theory might inspire for the classical version of Morse theory, where the Morse data of a critical point is determined by a single number, the Morse index.

In [Va] Vakhremeev proved the Morse theorems in the setting of manifolds with corners, a setting which strikes a nice balance between the simplicity of classical Morse theory on one hand, and the generality of Stratified Morse Theory on the other. Manifolds with corners are a class of stratified space that arise naturally in many applications, yet on these spaces the Morse data for an (essential) critical point still is determined by a single number.

Others have studied Morse Theory in similar settings, (e.g. Hamm ([Ha1], [Ha2]), Siersma ([Si]) and Braess ([Br]), but none of these make use of a global flow on manifolds with corners. The Morse theory for manifolds with corners developed in [Han] utilizes a modified gradient flow to prove the theorems in a more classical manner (e.g. as in [Mi]). This allows, as we shall see, the construction of a Morse

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complex with boundary maps determined by the trajectories connecting (essential) critical points.

1.1. **Setup and Definitions.** Let M be a compact n-dimensional manifold with corners endowed with a Riemannian metric. Then each $p \in M$ has a neighborhood diffeomorphic to an open subset of $[0, \infty)^n$.

We say that a diffeomorphism $\mathbf{x} : U \to [0, \infty)^j \times \mathbb{R}^{n-j}$ is a *standard coordinate chart* at p if U is an open subset containing p, $\mathbf{x}(p) = (0, \ldots, 0)$, and if $\frac{\partial}{\partial x_i} \in [0, \infty)^j \times \{0\}$ and $\frac{\partial}{\partial x_j} \in \{0\} \times \mathbb{R}^{n-j}$, then $\frac{\partial}{\partial x_i}$ is orthogonal to $\frac{\partial}{\partial x_j}$.

The number $j = j(p)$ for a standard coordinate chart at p is uniquely determined by p. M can be thought of as a stratified space. Each connected component of $\mathcal{E}_j = \{p \in M : j(p) = j\}$ is a stratum of dimension j.

We say that $f : M \to \mathbb{R}$ is a Morse function on M if the following hold:

1. If K is a stratum of M, and $p \in K$ is a critical point of $f|_K : K \to \mathbb{R}$, then either
   (a) p is a non-degenerate critical point of $f|_K : K \to \mathbb{R}$, i.e. the Hessian has non-zero determinant, or
   (b) the vector $-\nabla f(p)$ points into M.
2. If $p \in K$ is a critical point, then for any stratum $L \neq K$ with p in the closure of L, df_p is not identically zero on T_pL.
3. For any standard coordinate chart $\mathbf{x}$, whenever $-\nabla f(p)$ is tangent to a stratum $K \subseteq \partial M$ with $\frac{\partial}{\partial x_i} \perp K$ and $\frac{\partial f}{\partial x_i}(p) = 0$, the directional derivative of $\frac{\partial f}{\partial x_i}$ in the direction $-\nabla f(p)$ is not zero.

We say that p is an *essential critical point* of f if f satisfies condition (1a) at p, but not condition (1b). The *index* of an essential critical point p in a stratum K is equal to its index (in the classical sense) as a critical point of $f|_K$.

Suppose $f : M \to \mathbb{R}$ is a Morse function such that $f^{-1}(-\infty, r]$ is compact for each $r \in \mathbb{R}$. It is shown in [Han] that M is homotopy equivalent to a CW-complex with one cell of dimension λ for each essential critical point of index λ.

Let G be the *modified gradient vector field* of f on M, defined by projecting the vector $-\nabla f(p)$ onto the maximal stratum such that the resulting vector does not point outward from M. It is shown in [Han] that G induces a flow $\varphi : [0, \infty) \times M \to M$ and that the stationary points of this flow are exactly the essential critical points of f.

We can define the stable and unstable sets of an essential critical point p by

$$S(p) = \left\{ q \in M : \lim_{t \to \infty} \varphi(t, q) = p \right\},$$

and

$$U(p) = \left\{ q \in M : \exists \{q_k\}_{k=1}^{\infty} \subset M \text{ such that } \varphi(k, q_k) = q \text{ and } \lim_{k \to \infty} q_k = p \right\}.$$

1.2. **Results Proved.** Our goal is to describe the boundary maps of the CW-complex generated by f in terms of the trajectories of the flow φ. If we assume a Smale-like condition (Definition 1) for the function f then the number of trajectories τ connecting critical points p_2 and p_1 (where the index of p_2 is one greater than the index of p_1). Furthermore, if M is orientable, then we can associate to τ a sign $+1$ or -1. This sign is determined by comparing, via the flow φ, orientations on $U(p_1)$ and $U(p_2)$.

If P_1 and P_2 are the cells in the CW-complex corresponding to the critical points p_1 and p_2, then the degree of the attaching map of P_2 along P_1 is the sum of the signs of the trajectories from p_2 to p_1.

Now suppose M is an oriented manifold with corners and $f : M \to \mathbb{R}$ is a Morse function satisfying the conditions of Definition 1. Let $C_j(f)$ be the set of essential critical points of f with index j, and V_j the free Abelian group of formal $\mathbb{Z}$-linear combinations of the elements of $C_j(f)$. We can define $\partial_j : V_j \to V_{j-1}$ by setting

$$\partial_j(p) = \sum_{q \in C_{j-1}(f)} \text{degree}(p, q) q$$

for each $p \in C_j(f)$, and extending linearly to all of V_j. Finally, we can state the

Main Theorem. ***(Theorem 11)*** *The free Abelian groups V_j and maps ∂_j form a chain complex whose homology groups are identical to the $\mathbb{Z}$-homology groups of the topological space M.*

2. The Attaching Map of a Cell in the Morse Complex

If p is an essential critical point with index λ, we can choose a coordinate chart such that

$$f(\mathbf{x}) = c - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_{n-j}^2 + x_{n-j+1} + \cdots + x_n$$

We can define, for ε sufficiently small, a region H with the property that $M_{c-\varepsilon} \cup H$ is homotopy equivalent to both $M_{c+\varepsilon}$ and $M_{c-\varepsilon} \cup e^\lambda$. To do so we first choose a smooth function $\mu : \mathbb{R} \to \mathbb{R}$ so that $\mu(0) > \varepsilon$, $\mu(r) = 0$ for $r > 2\varepsilon$, and $-1 < \mu' \leq 0$. Then we set

$$F(\mathbf{x}) = f(\mathbf{x}) - \mu(x_1^2 + \cdots + x_\lambda^2 + 2(x_{\lambda+1}^2 + \cdots + x_{n-j}^2) + 2(x_{n-j+1} + \cdots + x_n))$$

Finally, we define $H = \overline{F^{-1}(-\infty, c - \varepsilon] - M_{c-\varepsilon}}$.

Let p_1 and p_2 be essential critical points with indices λ_1 and λ_2 respectively. In addition, suppose that $f(p_1) = c_1$ and $f(p_2) = c_2$, where $c_1 < c_2$, and that there are no other critical points in $f^{-1}([c_1, c_2])$. Then we can choose ε sufficiently small that the set

$$U_\varepsilon(p_2) = U(p_2) - M_{c_2-\varepsilon}$$

is homeomorphic to e^{λ_2}, a cell with dimension λ_2.

We can then think of $M_{c_2-\varepsilon} \cup U(p_2)$ as $M_{c_2-\varepsilon}$ with a λ_2-cell attached via the inclusion map

$$\sigma : \partial U_\varepsilon(p_2) \hookrightarrow M_{c_2-\varepsilon}.$$

Now, the flow φ gives rise to a family of homotopy equivalences which can be thought of in two ways:

(1) $\varphi(t, \cdot)$ gives a deformation retraction of $M_{c_2-\varepsilon} \cup U(p_2)$ onto

$$\varphi(t, M_{c_2-\varepsilon} \cup U(p_2)) = \varphi(t, M_{c_2-\varepsilon}) \cup U(p_2).$$

(2) $\varphi(t, \cdot)$ gives a deformation retraction of $M_{c_2-\varepsilon} \cup_\sigma e^{\lambda_2}$ onto

$$\varphi(t, M_{c_2-\varepsilon}) \cup_{\varphi(t,\cdot)\circ\sigma} e^{\lambda_2}.$$

In the first case, we allow the whole space to flow by φ. The equality follows from the φ-invariance of $U(p_2)$. In the second case, we simply allow the image of the attaching map σ_2 to flow by φ. (c.f. Lemma 3.7 in [Mi].) The distinction is important, since although $U_\varepsilon(p_2)$ is homeomorphic to e^{λ_2}, the unstable set $U(p_2)$ may not be. This is a notable difference from the classical theory.

Now we prove the following

Lemma 1. *Let p_1 and p_2 be two essential critical points of f such that $f(p_1) = c_1$, $f(p_2) = c_2$, $c_1 < c_2$ and that there are no other critical points in $f^{-1}([c_1 - \varepsilon, c_2])$. Then*

$$M_{c_2-\varepsilon} \cup U(p_2) \simeq M_{c_1-\varepsilon} \cup H \cup U(p_2)$$

Proof. For each $q \in M_{c_2-\varepsilon}$, define

$$t_q^{(1)} = \inf\left\{t \in \mathbb{R}^+ : \varphi(t,q) \in M_{c_1-\varepsilon} \cup H\right\}$$

The function $q \mapsto t_q^{(1)}$ is continuous, and $t_q^{(1)}$ is finite for each $q \in M_{c_2-\varepsilon}$. Moreover, since $M_{c_2-\varepsilon}$ is compact, $T_1 = \sup\{t_q^{(1)} : q \in M_{c_2-\varepsilon}\}$ is finite.

Now we can define the desired homotopy $\Phi_2 : [0,1] \times M_{c_2-\varepsilon} \cup U(p_2) \to M_{c_1-\varepsilon} \cup H \cup U(p_2)$ by

$$\Phi_2(s,q) = \begin{cases} \varphi(T_1 s, q) & T_1 s \le t_q^{(1)} \\ \varphi(t_q^{(1)}, q) & T_1 s \ge t_q^{(1)} \end{cases}$$

q.e.d.

There is no reason, in principle, that the ε used to define $U_\varepsilon(p_2)$ must be the same as the ε in Lemma 1. They can be chosen to be the same, however, and this serves to simplify the notation in the remainder of our argument.

Now lets view this in terms of attaching maps. We have the inclusion map $\sigma_2 : \partial U_\varepsilon(p_2) \to M_{c_2-\varepsilon}$. Then we can use the homotopy Φ_2 to define an attaching map

$$\Phi_2(1,\cdot) \circ \sigma_2 : \partial U_\varepsilon(p_2) \to M_{c_1-\varepsilon} \cup H.$$

Now, identifying $U_\varepsilon(p_2)$ with e^{λ_2}, we find that

$$M_{c_2-\varepsilon} \cup U(p_2) \simeq M_{c_1-\varepsilon} \cup H \cup_{\Phi_2(1,\cdot)\circ\sigma_2} e^{\lambda_2}.$$

Note that the homotopy Φ_2 preserves the trajectories of φ. This will be an important fact when we describe the degree of the attaching maps in Section 3.

Lemma 2. *There is an attaching map $\Sigma_2 : \partial e^{\lambda_2} \to M_{c_1-\varepsilon} \cup U(p_1)$ such that*

$$M_{c_1-\varepsilon} \cup U(p_1) \cup_{\Sigma_2} e^{\lambda_2} \simeq M_{c_1-\varepsilon} \cup H \cup U(p_2).$$

Proof. Choose a coordinate system near p_1 in which

$$f(\mathbf{x}) = f(p_1) - x_1^2 - \cdots - x_\lambda^2 + x_{\lambda+1}^2 + \cdots + x_{n-j}^2 + x_{n-j+1} + \cdots + x_n.$$

Then the stable set of p_1 corresponds to $\{(x_1, \ldots, x_n) | x_1 = \cdots = x_\lambda = 0\}$. The unstable set of p_1 corresponds to $\{(x_1, \ldots, x_n) | x_{\lambda+1} = \cdots = x_n = 0\}$.

Consider the function $g(\mathbf{x}) = f(p_1) + x_{\lambda+1}^2 + \cdots + x_{n-j}^2 + x_{n-j+1} + \cdots + x_n$. This function is constant on the unstable set, with a value of $f(p_1)$. For q outside the stable set, $f(q) > f(p_1)$. Moreover, the function g agrees with f on the stable set.

Let X denote the modified (negative) gradient vector field of the function g. Then $X(q) = G(q)$ for $q \in S(p_1)$, and $X(q) = 0$ for $q \in U(p_1)$. The flow of X is directed toward $U(p_1)$. Let ψ denote the flow of the vector field $\frac{X}{\|X\|}$, and note that ψ preserves the trajectories that have p_1 as a lower endpoint.

For each q in H, we can set $t_q^{(2)} = \inf\{t \in \mathbb{R}^+ : \psi(t,q) \in M_{c_1-\varepsilon} \cup U(p_1)\}$, and for $q \in M_{c_1-\varepsilon}$ set $t_q^{(2)} = 0$. Then define $T_2 = \sup\{t_q^{(2)} : q \in H\}$. Then the map $\Psi_2 : [0,1] \times M_{c_1-\varepsilon} \cup H \to M_{c_1-\varepsilon} \cup U(p_1)$ given by

$$\Psi_2(s,q) = \begin{cases} \psi(T_2 s, q) & T_2 s \leq t_q^{(2)} \\ \psi(t_q^{(2)}, q) & T_2 s \geq t_q^{(2)} \end{cases}$$

gives a homotopy equivalence $\Psi_2(1,\cdot) : M_{c_1-\varepsilon} \cup H \to M_{c_1-\varepsilon} \cup U(p_1)$. Consequently,

$$\begin{aligned} M_{c_1-\varepsilon} \cup H \cup U(p_2) &\simeq M_{c_1-\varepsilon} \cup H \cup_{\Phi_2(1,\cdot)\circ\sigma_2} e^{\lambda_2} \\ &\simeq M_{c_1-\varepsilon} \cup U(p_1) \cup_{\Psi_2(1,\cdot)\circ\Phi_2(1,\cdot)\circ\sigma_2} e^{\lambda_2}. \end{aligned}$$

The first homotopy equivalence follows from Lemma 1.

Finally, we define $\Sigma_2 = \Psi_2(1,\cdot) \circ \Phi_2(1,\cdot) \circ \sigma_2$. q.e.d.

It is important to note that, although the homotopy Ψ_2 does not preserve the trajectories of φ, it does preserve those trajectories that lie within the stable set $S(p_2)$. This will be important to us in determining the degree of the attaching map Σ_2.

The homotopy equivalence of M with the desired CW-complex can be defined inductively. If $p_1, p_2, \ldots, p_m$ are the essential critical points of M, each with index λ_i and labeled such that $f(p_1) < f(p_2) < \ldots$, then $c_1 = f(p_1)$ must be an absolute minimum, and $M_{c_1+\varepsilon}$ is homotopy equivalent to the 0-cell, $U_\varepsilon(p_1)$. In fact the map $\Psi_2(1,\cdot)$ is a homotopy equivalence.

Now, as in Lemma 1 and Lemma 2, we consider the inclusion map $\sigma_2 : \partial U_\varepsilon(p_2) \hookrightarrow M_{c_2-\varepsilon}$. Viewing $U_\varepsilon(p_2)$ as a λ_2-cell, and using the attaching map $\Sigma_2 = \Psi_2(1,\cdot) \circ \Phi_2(1,\cdot) \circ \sigma_2$, of $U_\varepsilon(p_2)$ onto $U_\varepsilon(p_1)$ we see that

$$M_{c_2+\varepsilon} \simeq U_\varepsilon(p_1) \cup_{\Sigma_2} U_\varepsilon(p_2).$$

Again, we view $U_\varepsilon(p_3)$ as λ_3-cell. It takes only a slight extension to define the attaching map for this cell. The map $\widetilde{\Sigma}_3 = \Psi_3(1,\cdot) \circ \Phi_3(1,\cdot) \circ \sigma_3$ gives the appropriate attaching map of $\partial U_\varepsilon(p_3)$ onto $M_{c_2-\varepsilon} \cup U(p_2) \simeq M_{c_2-\varepsilon} \cup_{\sigma_2} U_\varepsilon(p_2)$. Note that $\Psi_2(1,\cdot) \circ \Phi_2(1,\cdot)$ provides a homotopy equivalence $M_{c_2-\varepsilon} \cup_{\sigma_2} U_\varepsilon(p_2) \to U_\varepsilon(p_1) \cup_{\Sigma_2} U_\varepsilon(p_2)$. Then the desired attaching map is given by

$$\Sigma_3 = \Psi_2(1,\cdot) \circ \Phi_2(1,\cdot) \circ \Psi_3(1,\cdot) \circ \Phi_3(1,\cdot) \circ \sigma_3.$$

It follows that

$$M_{c_3+\varepsilon} \simeq U_\varepsilon(p_1) \cup_{\Sigma_2} U_\varepsilon(p_2) \cup_{\Sigma_3} U_\varepsilon(p_3).$$

In general, for each critical point p_k we define

$$\Sigma_k = \Psi_2(1,\cdot) \circ \Phi_2(1,\cdot) \circ \ldots \Psi_k(1,\cdot) \circ \Phi_k(1,\cdot) \circ \sigma_k.$$

These maps gives a homotopy equivalence

$$M \simeq U_\varepsilon(p_1) \cup_{\Sigma_2} U_\varepsilon(p_2) \cup \cdots \cup_{\Sigma_m} U_\varepsilon(p_m).$$

We shall see that under appropriate conditions (Definition 1), this cellular complex is in fact a CW-complex.

3. The Degree of an Attaching Map

Our goal is to show that, as in classical Morse theory, the degree of this attaching map can be determined by counting the trajectories connecting essential critical points. The difficulty is that, although these attaching maps are continuous, they are in general not smooth. Lemmas 6, 7 and 8 show that the maps possess enough regularity that we can establish a Smale-like transversality condition (Definition 1). Using this definition, we can show that a finite number of trajectories connect critical points whose indices differ by one. Finally we define a degree to each such pair.

Lemma 3. *Let K be a stratum with dimension k and L be a stratum in the closure of K with dimension $k-1$. Let $p \in K$ and suppose that $\varphi(s,p) \in K$ for $s < t_p$ and $\varphi(s,p) \in L$ for $t_p \leq s \leq T$. Then there is a neighborhood $U \subset K$ of p such that $\varphi(T,\cdot) : U \to M$ is smooth.*

Proof. Choose a coordinate system near $\varphi(t_p,p)$ such that for $q \in K$, $x_{k+1}(q) = \cdots = x_n(q) = 0$ and for $q \in L$, $x_k(q) = \cdots = x_n(q) = 0$. We know that the kth coordinate satisfies $\varphi_k(t_p,p) = 0$, and in addition

$$\left.\frac{\partial \varphi_k}{\partial t}\right|_{(t_p,p)} \neq 0,$$

by the definition of Morse function.

From the Implicit Function Theorem, it follows that there is an open set $U \subset K$ containing p and an open set $V \subset \mathbb{R}$ such that for each $q \in U$ there is a unique $g(q) \in V$ such that $\varphi_k(g(q),q) = 0$. Moreover, the function g is smooth.

We can use this function g to define two additional functions: $\gamma : U \to \mathbb{R} \times U$ given by $\gamma(q) = (g(q),q)$ and $\tau : \gamma(U) \to \mathbb{R} \times M$ given by $\tau(t,q) = (T-t, \varphi(t,q))$. Since φ is smooth whenever it remains in a single stratum, these are themselves smooth functions. We can write

$$\varphi(T,q) = (\varphi \circ \tau \circ \gamma)(q),$$

a composition of smooth functions, so $\varphi(T,\cdot) : U \to M$ is smooth.

q.e.d.

Lemma 4. *Let K be a stratum with dimension $k-1$ in the closure of a stratum L with dimension k. Let $p \in K$ and suppose that $\varphi(s,p) \in K$ for $s \leq t_p$ and $\varphi(s,p) \in L$ for $t_p < s \leq T$. Then there is a neighborhood $U \subset K$ of p such that $\varphi(T,\cdot) : U \to M$ is smooth.*

Proof. Choose a coordinate system near $\varphi(t_p,p)$ such that for $q \in K$, $x_k(q) = \cdots = x_n(q) = 0$ and for $q \in L$, $x_{k+1}(q) = \cdots = x_n(q) = 0$. Let $h(s,q) = (-\nabla f)_k(\varphi(s,q))$. Then h is a smooth real valued function. Since the trajectory $\varphi(\cdot,p)$ moves from K to L at time t_p, we must have $h(t_p,p) = 0$.

By the definition of a Morse function, we know that

$$\left.\frac{\partial h}{\partial t}\right|_{(t_p,p)} \neq 0.$$

It then follows from the Implicit Function Theorem, that there is an open set $U \subset K$ containing p and an open subset $V \subset \mathbb{R}$ such that for each $q \in U$ there is a unique $g(q) \in V$ such that $h(g(q),q) = 0$. Moreover, the function g is smooth.

As before, we can define smooth functions $\gamma : U \to \mathbb{R}\times U$ given by $\gamma(q) = (g(q), q)$ and $\tau : \gamma(U) \to \mathbb{R} \times M$ given by $\tau(t, q) = (T - t, \varphi(t, q))$ and write

$$\varphi(T, q) = (\varphi \circ \tau \circ \gamma)(q),$$

Again, this is a composition of smooth functions, and so $\varphi(T, \cdot) : U \to M$ is smooth. q.e.d.

Now let p_2 be an essential critical point with index $\lambda_2 = j$ and p_1 an essential critical point with index $\lambda_1 = j - 1$. Suppose that $S(p_1) \cap U(p_2) \neq \emptyset$. Choose a

$$q \in S(p_1) \cap U(p_2) \cap W.$$

Then there is a $q_1 \in U_\varepsilon(p_2)$ and a time T such that $q = \varphi(T, q_1)$.

Lemma 5. *If the trajectory $\varphi(s, q_1)$ only ever changes between strata whose difference in dimension is 1, then there is a neighborhood V in the stratum containing q_1 such that $\varphi(T, \cdot) : V \to M$ is smooth.*

Proof. The result follows from repeated application of Lemma 6 and Lemma 7. Since $f^{-1}(-\infty, r]$ is compact for each choice of r, there can be only finitely many stratum changes ([Han]). For each stratum change, $i = 1, \ldots, N$, we get a corresponding neighborhood V_i. Since there are finitely many, we may choose

$$V = V_1 \cap \cdots \cap V_N.$$

q.e.d.

Definition 1. We say that the Morse function f is a *Morse-Smale function* if for each pair of critical points, p and p' with $f(p) > f(p')$, and for ε sufficiently small the parametrized space $U(p)$ and the set $S_\varepsilon(p') = S(p') \cap f^{-1}((-\infty, f(p') + \varepsilon])$ intersect transversely.

Let p_i and p_j be two critical points with indices λ_i and λ_j, respectively, satisfying $\lambda_j - \lambda_i = 1$. Let $\mathfrak{T}$ be the set of trajectories from p_j to p_i. We wish to show that the transversality condition in Definition 1 together with the compactness of M ensure that $\mathfrak{T}$ is a finite set.

We begin with the following

Lemma 6. *Suppose that $\widetilde{M}$ is an m-manifold, $\widetilde{S}$ a k-dimensional regular submanifold. Let $\widetilde{U}$ be an $(m - k)$-manifold, and $\Phi : \widetilde{U} \to \widetilde{M}$ a smooth map such that $\widetilde{S} \pitchfork \Phi(\widetilde{U})$. Then $\Phi^{-1}(\widetilde{S} \cap \Phi(\widetilde{U}))$ consists of isolated points in $\widetilde{U}$.*

Proof. First, since $\dim(\widetilde{S}) + \dim(\Phi(\widetilde{U})) = k + (m - k) = m$, the transverse intersection of $\widetilde{S}$ and $\Phi(\widetilde{U})$ is zero dimensional. Let $q \in \widetilde{U}$ such that $\Phi(q) \in \widetilde{S}$. We will show that there is a neighborhood N_q of q such that $\Phi^{-1}(\widetilde{S}) \cap N_q = \{q\}$.

let $\mathbf{x} = (x^1, \ldots, x^m)$ be a coordinate chart on $\widetilde{M}$ such that $\mathbf{x}(\Phi(q)) = (0, \ldots, 0)$, and $(x^1, \ldots, x^k)$ gives a coordinate chart on $\widetilde{S}$. Then for $p \in \widetilde{S}$

$$x^i(p) = 0, \quad \text{for } i = k + 1, \ldots, m.$$

Let $\bar{\mathbf{x}} = (x^{k+1}, \ldots, x^m)$. We want to show that $\tilde{\mathbf{x}} = \bar{\mathbf{x}} \circ \Phi$ is a coordinate chart on $\widetilde{U}$. It suffices to show that if $\mathbf{y}$ is a coordinate chart on $\widetilde{U}$, then $D(\tilde{\mathbf{x}} \circ \mathbf{y}^{-1})$ is non-singular.

Since $\widetilde{S} \pitchfork \Phi(\widetilde{U})$, the vectors

$$\left\{ \frac{\partial}{\partial x^1}|_{\Phi(q)}, \dots, \frac{\partial}{\partial x^k}|_{\Phi(q)}, \Phi_* \left(\frac{\partial}{\partial y^{k+1}|_q} \right), \dots, \Phi_* \left(\frac{\partial}{\partial y^n|_q} \right) \right\}$$

form a basis for $T_{\Phi(q)}(\widetilde{M})$, as do $\left\{ \frac{\partial}{\partial x^1}|_{\Phi(q)}, \dots, \frac{\partial}{\partial x^m}|_{\Phi(q)} \right\}$. We can thus write

$$\Phi_* \left(\frac{\partial}{\partial y^i} \right) = \sum_{j=1}^{n} \alpha_i^j \frac{\partial}{\partial x^j}.$$

Then the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ \alpha_{k+1}^1 & & \cdots & \alpha_{k+1}^m \\ \vdots & & & \vdots \\ \alpha_m^1 & & \cdots & \alpha_m^m \end{bmatrix} = \left[\begin{array}{c|ccc} I_k & & 0 & \\ \hline & \alpha_{k+1}^{k+1} & \cdots & \alpha_{k+1}^m \\ * & \vdots & \ddots & \vdots \\ & \alpha_m^{k+1} & \cdots & \alpha_m^m \end{array} \right] = \left[\begin{array}{c|c} I_k & 0 \\ \hline * & M \end{array} \right]$$

is non-singular (having linearly independent rows). It follows that the matrix M is non-singular. But M is the transpose of the matrix for

$$D(\bar{\mathbf{x}} \circ (\Phi \circ \mathbf{y}^{-1})) = D((\bar{\mathbf{x}} \circ \Phi) \circ \mathbf{y}^{-1}) = D(\tilde{\mathbf{x}} \circ \mathbf{y}^{-1}),$$

evaluated at $\mathbf{y}(q)$, so $D(\tilde{\mathbf{x}} \circ \mathbf{y}^{-1})$ is non-singular in a neighborhood of $\mathbf{y}(q)$.

Now, since $\tilde{\mathbf{x}}$ is a coordinate chart in a neighborhood of q, there is an open set N_q of q such that for $q' \in N_q$ with $q' \neq q$, $\tilde{\mathbf{x}}(q') \neq (0, \dots, 0)$. So $\Phi(q') \notin S$.

q.e.d.

Lemma 7. *Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a manifold with corners M. Let p_i and p_j be two critical points of f with indices λ_i and λ_j respectively, satisfying $\lambda_j - \lambda_i = 1$. Then the set $\mathfrak{T}$ of trajectories from p_j to p_i is finite.*

Proof. Let $c_i = f(p_i)$ and $c_j = f(p_j)$. Define $\widetilde{U} = \partial U_\varepsilon(p_j)$. Then $\widetilde{U}$ is diffeomorphic to $S^{\lambda_j - 1}$. For each element of $\tau \in \mathfrak{T}$ there is exactly one $q_\tau \in \widetilde{U} \cap \tau$.

Suppose that $\mathfrak{T}$ is an infinite set. Then, since $\widetilde{U}$ is compact, there is a convergent sequence $\{q_i\}_{i=1}^\infty$ such that $q_i \in \widetilde{U} \cap \tau_i$ for some $\tau_i \in \mathfrak{T}$. Let $q_0 = \lim_{i \to \infty} q_i$.

We wish to show that $q_0 = \widetilde{U} \cap \tau_0$ for some $\tau_0 \in \mathfrak{T}$. Choose T such that for $p = \varphi(T, q_0)$, $f(p) < c_i + \varepsilon$. Then for i sufficiently large $f(\varphi(T, q_i)) < c_i + \varepsilon$, and since for each i $\lim_{t \to \infty} \varphi(t, q_i) = p_i$, $\varphi(T, q_i) \in S_\varepsilon(p_i)$ for i sufficiently large. Now, φ is continuous, and $S_\varepsilon(p_i)$ is closed, so $p = \lim_{i \to \infty} \varphi(T, q_i) \in S_\varepsilon(p_i)$. It follows that q_0 lies on some trajectory connecting p_j to p_i.

Let $c = f(p)$. Let $\widetilde{M} = f^{-1}(c)$, $\widetilde{S} = S_\varepsilon(p_i) \cap \widetilde{M}$ and let U_0 be a neighborhood of q_0 on which $\varphi(T, \cdot)$ is smooth. For each $q \in U_0$ there is a trajectory of φ containing q. We will denote this trajectory by γ_q. For a given $q \in U_0$, let $\tilde{q} = \gamma_q \cap \widetilde{U}$, $\bar{q} = \gamma_q \cap \varphi(T, \cdot)^{-1}(S)$, and let $t_q \in \mathbb{R}$ such that

$$\begin{aligned} \varphi(t_q, \tilde{q}) = \bar{q} \quad & \text{if } f(\tilde{q}) \geq f(\bar{q}) \\ \varphi(-t_q, \bar{q}) = \tilde{q} \quad & \text{if } f(\tilde{q}) \leq f(\bar{q}). \end{aligned}$$

The maps $q \mapsto \tilde{q}$, $q \mapsto \bar{q}$ and $q \mapsto t_q$ are all smooth, and so the function $\Phi_0 : U_0 \to M$ given by $\Phi_0(q) = \varphi(T + t_q, q)$ is smooth. Moreover, $\Phi = \Phi_0|_{\widetilde{U} \cap U_0} : \widetilde{U} \cap U_0 \to \widetilde{M}$.

We can let $k = \dim(\widetilde{S}) = n - \lambda_i - 1$ and $m = \dim(\widetilde{M}) = n-1$. Then $\dim(\widetilde{U}) = \lambda_j - 1 = m - k$. Choose a coordinate chart at $p \in \widetilde{M}$, $\mathbf{x} = (x^1, \dots, x^m)$, such that $(x^1, \dots, x^k)$ is a coordinate chart on $\widetilde{S}$ and

$$x^i(p') \neq 0 \quad \text{for } p' \notin \widetilde{S} \text{ and } i > k.$$

This can be done in such a way that $\frac{\partial}{\partial x^{k+1}}|_p$ is tangent to the trajectory τ_0.

Now choose a coordinate chart $\mathbf{y} = (y^{k+1}, \dots, y^n)$ on $U_\varepsilon(p_j)$, such that $\mathbf{y} = (y^{k+1}, \dots, y^m)$ is a coordinate chart on $\widetilde{U}$ and $\frac{\partial}{\partial y^n}|_{q_0}$ is tangent to the trajectory τ_0. Then

$$\Phi_*(\frac{\partial}{\partial y^i}|_{q_0}) \in T_p\widetilde{M} \text{ for } i = k+1, \dots, m.$$

In addition, $\Phi_*(\frac{\partial}{\partial y^n}|_{q_0})$ will be tangent (at p) to the trajectory τ_0.

Since f is a Morse-Smale function, the vectors

$$\left\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^{k+1}}|_p, \Phi_*(\frac{\partial}{\partial y^{k+1}}|_{q_0}), \dots, \Phi_*(\frac{\partial}{\partial y^n}|_{q_0})\right\}$$

span T_pM. Since the vectors $\frac{\partial}{\partial x^{k+1}}|_p$ and $\Phi_*(\frac{\partial}{\partial y^n}|_{q_0})$ are tangent to $T_p\tau_0$, and the remaining $m = \dim(\widetilde{M})$ vectors are all in $T_p\widetilde{M}$,

$$\left\{\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^{k}}|_p, \Phi_*(\frac{\partial}{\partial y^{k+1}}|_{q_0}), \dots, \Phi_*(\frac{\partial}{\partial y^m}|_{q_0})\right\}$$

form a basis for $T_p\widetilde{M}$. The first k vectors of this basis are a basis of $T_p\widetilde{S}$, and the remaining $m-k$ vectors are a basis for $T_p\Phi(\widetilde{U})$. It follows that

$$\widetilde{S} \pitchfork \Phi(\widetilde{U}).$$

By Lemma 9, $\Phi^{-1}(\widetilde{S} \cap \Phi(\widetilde{U}))$ consists of isolated points. Note that for sufficiently large i, $q_i \in \Phi^{-1}(\widetilde{S} \cap \Phi(\widetilde{U}))$ and in addition $q_0 \in \Phi^{-1}(\widetilde{S} \cap \Phi(\widetilde{U}))$. This, however, contradicts the fact that $\lim_{i\to\infty} q_i = q_0$. Consequently, the number of trajectories connecting p_j to p_i must be finite. q.e.d.

Now assume that M is an oriented manifold. At each essential critical point p, the tangent space T_pM can be decomposed into a stable space and an unstable space, $T_pM = E^+(p) \oplus E^-(p)$. Choose an orientation for the unstable space $E^-(p)$. This induces an orientation on $E^+(p)$ in the following way:

Choose a basis $v_1, \dots, v_{\lambda(p)} \in E^-(p)$ which represents the orientation on $E^-(p)$. Then choose $v_{\lambda(p)+1}, \dots, v_n \in E^+(p)$ such that $v_1, \dots, v_n$ is a basis which represents the orientation on T_pM. The vectors $v_{\lambda(p)+1}, \dots, v_n$ determine an orientation for $E^+(p)$.

Now for critical points p_i and p_j, the orientation on $E^+(p_i)$ determines an orientation on $S_\varepsilon(p_i)$, which in turn determines an orientation on $T_qS(p_i)$ for $q \in S(p_i)$. In a similar way, the choice of orientation on $E^-(p_j)$ induces an orientation on $T_{q'}U(p_j)$ for $q' \in U(p_j)$.

Note that if the function $f : M \to \mathbb{R}$ is a Morse-Smale function and $\varphi(T, q') = q$, then the map $\varphi(T, \cdot)_*$ restricted to $U(p_j)$ has full rank at $\varphi(T, \cdot)^{-1}(q)$.

Now the map $\varphi(T, \cdot)_*$ pushes forward the orientation on $T_{q'}U(p_j)$ to give an orientation on $T_q\varphi(T, U(p_j))$. Since $\dim(\varphi(T, U(p_j))) = j$ and $\dim(S_\varepsilon(p_i)) = n - j + 1$, these two sets intersect in the one dimensional trajectory through q. We can choose a basis for $T_q\varphi(T, U(p_j))$ which represents the induced orientation and such that the last vector is $\frac{d}{dt}|_T\varphi(t,q_1)$. We also may choose a basis representing the orientation on $T_qS(p_i)$, with $\frac{d}{dt}|_T\varphi(t,q_1)$ as its first vector. When combined, these bases give an orientation for T_qM. If this orientation agrees with the chosen orientation on M then we say the sign of the trajectory from p_j to p_i through q_1 is $+1$. Otherwise, the sign is -1.

Now if $\mathfrak{T}$ is the set of trajectories from p_j to p_i, Lemma 10 ensures that $\mathfrak{T}$ is a finite set. So, we can define

$$\text{degree}(p_j,p_i)=\sum_{\tau\in\mathfrak{T}}\text{sign}(\tau).$$

This degree, we shall see, is the degree of the attaching map Σ_j at p_i.

4. The Morse Complex for a Morse Function on a Manifold with Corners

Theorem 8. *Let M be a compact orientable manifold with corners, and $f : M\to\mathbb{R}$ be a Morse-Smale function satisfying the conditions of Lemma 8. Let $C_j(f)$ be the set of essential critical points of f with index j, and V_j the free Abelian group of formal $\mathbb{Z}$-linear combinations of the elements of $C_j(f)$. For each $j=1,\ldots,n=\dim(M)$ define a map $\partial_j : V_j\to V_{j-1}$ by setting*

$$\partial_j(p)=\sum_{q\in C_{j-1}(f)}\text{degree}(p,q)q$$

for each $p\in C_j(f)$, and extending linearly to all of V_j. For $j=0$, set $\partial_0(p)=0$

The free Abelian groups V_j and maps ∂_j form a chain complex whose homology groups are identical to the $\mathbb{Z}$-homology groups of the topological space M.

Proof. Let P_k denote the cell $U_\varepsilon(p_k)$. We know that M is homotopy equivalent to the cellular complex

$$X=P_1\cup_{\Sigma_2}P_2\cup\cdots\cup_{\Sigma_m}P_m.$$

Let $C_k(X)$ be the set of cells in X having dimension k. Let W_k be the free Abelian group of formal $\mathbb{Z}$-linear combinations of the elements of $C_k(X)$. Note that W_k is naturally isomorphic to V_k, since there is a one-to-one correspondence between cells $P_j\in C_k(X)$ and critical points $p_j\in C_k(f)$.

We define maps $\partial_k^{(W)} : W_k\to W_{k-1}$ by setting

$$\partial_k^{(W)}(P_j)=\sum_{P_i\in C_{k-1}(X)}\deg(\Sigma_j,P_i)$$

for each $P_j\in C_k(X)$, where Σ_j is the attaching map for the cell P_j. The degree $\deg(\Sigma_j,P_i)$ is determined by choosing a $q\in P_i$ such that $\Sigma_j(\partial P)$ intersects P_i transversely at q, and then adding the number of points in $\Sigma_j^{-1}(q)$ at which Σ_j is orientation preserving, and subtracting the number of points in $\Sigma_j^{-1}(q)$ at which Σ_j is orientation reversing. Given $P_j\in C_k(X)$ and $P_i\in C_{k-1}(X)$ we must determine this degree.

Recall that Σ_j was defined by

$$\Sigma_j=\Psi_2(1,\cdot)\circ\Phi_2(1,\cdot)\circ\cdots\circ\Psi_j(1,\cdot)\circ\Phi_j(1,\cdot)\circ\sigma_j,$$

which attaches $U_\varepsilon(p_j)$ to $P_1\cup_{\Sigma_2}P_2\cup\cdots\cup_{\Sigma_{j-1}}P_{j-1}$ This map can be thought of as a sequence of homotopy equivalences, one for each of the critical points $p_1,\ldots,p_{j-1}$.

We can define

$$\Sigma_{ij} = \Psi_{i+1}(1,\cdot) \circ \Phi_{i+1}(1,\cdot) \circ \cdots \circ \Psi_j(1,\cdot) \circ \Phi_j(1,\cdot) \circ \sigma_j.$$

Then $\Sigma_{ij} : \partial U_\varepsilon(p_j) \to M_{c_i-\varepsilon} \cup_{\Sigma_i} U(p_i) \cup \cdots \cup_{\Sigma_{j-1}} U_\varepsilon(p_{j-1})$. We have the relationship

$$\deg(\Sigma_j, P_i) = \deg(\Sigma_{ij}, U_\varepsilon(p_i)).$$

The map Σ_{ij} is defined by a sequence of flows. They alternate between a Φ_k which follows the φ-trajectories, and a Ψ_k which does not. Note, though, that the flow Ψ_k deviates from the φ-trajectories only in a neighborhood of each critical point p_k. If ε is chosen to be small enough, we can ensure that these neighborhoods do not intersect any of the φ-trajectories connecting p_j to p_i (except for the region H around p_i). But the flow Ψ_{i+1} on H preserves $S(p_i)$, so $\Sigma_{ij}^{-1}(p_i)$ consists of those points in $\partial U_\varepsilon(p_j)$ that lie on trajectories connecting p_j to p_i. Since f is a Morse-Smale function $\Sigma_{ij}(\partial U_\varepsilon(p_j))$ will intersect $\partial U_\varepsilon(p_i)$ transversely at p_i.

Consider a point $q \in \partial U_\varepsilon(p_j) \cap \tau$ for some trajectory connecting p_j to p_i. The map Σ_{ij} is orientation preserving at q if $\text{sign}(\tau) = +1$, and orientation reversing if $\text{sign}(\tau) = -1$. It follows that

$$\deg(\Sigma_j, P_i) = \deg(\Sigma_{ij}, p_i) = \text{degree}(p_j, p_i).$$

The above argument also shows that if

$$\text{index}(p_i) \geq \text{index}(p_j)$$

then no trajectories of φ connect p_j to p_i, and so

$$\Sigma_j(\partial U_\varepsilon(p_j)) \cap U_\varepsilon(p_i) = \emptyset$$

(assuming ε is sufficiently small). So the attaching maps carry the boundary of each cell to a collection of lower dimensional cells. It follows that the cellular complex X is a CW-complex. The $\mathbb{Z}$-homology groups of X are the homology groups of the chain complex

$$0 \longleftarrow W_0 \xleftarrow{\partial_1^{(W)}} W_1 \xleftarrow{\partial_2^{(W)}} \ldots \xleftarrow{\partial_n^{(W)}} W_n.$$

But since $W_k \cong V_k$ and $\deg(\Sigma_j, P_i) = \text{degree}(p_j, p_i)$, the homology groups of the chain complex

$$0 \longleftarrow V_0 \xleftarrow{\partial_1} V_1 \xleftarrow{\partial_2} \ldots \xleftarrow{\partial_n} V_n$$

are the same as the $\mathbb{Z}$-homology groups of X. Since homology groups are a homotopy invariant, and X is homotopy equivalent to M, the result follows.

q.e.d.

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