Exam #1 Review

The exam will be closed book and notes; only the following calculators will be permitted: TI-30X IIS, TI-30X IIB, TI-30Xa.

1. **(25 points)**

   Consider a simple financial model with two time $t = 0, 1$, two stocks S^1 and S^2 and a one period interest rate of $r = .10$. The initial prices for the stocks are $S_0^1 = S_0^2 = \$20$. There are three possible outcomes for the stock prices, $\{\omega_1, \omega_2, \omega_3\}$. The stock prices in each case are

   $$S_1^1(\omega_1) = 24, \quad S_1^1(\omega_2) = 18, \quad S_1^1(\omega_3) = 16,$$

   and

   $$S_1^2(\omega_1) = 24, \quad S_1^2(\omega_2) = 24, \quad S_1^2(\omega_3) = 8.$$

   Consider a derivative security V whose value at time $t = 1$ is given by

   $$V_1(\omega_i) = |S_1^1(\omega_i) - S_1^2(\omega_i)|.$$

   (a) Explain, without constructing a replicating portfolio, why $\$0 < V_0 < \$\frac{8}{1.10}$.

   (b) Find a replicating strategy, and use it to determine V_0.

2. **(25 points)**

   Suppose that between $t = 0$ and $t = 1$, you can borrow or invest any number of dollars at the one period rate of $r^{\$} = .06$. You can also borrow or invest any number of British pounds at the one period rate of $r^{£} = .04$. At $t = 0$ the value in dollars of one pound is $E_{\$}^{£} = 2$.

   A stock S is listed on an American exchange, and can be purchased at $t = 0$ for $S_0 = \$50$ per share.

   Consider a forward contract that at time $t = 1$ requires its holder to receive one share of S in exchange for a payment of $£K$, with the payment determined at $t = 0$. Find a value for K that makes the value of the contract at $t = 0$ equal to zero. Explain your reasoning with reference to a replicating portfolio.

3. **(25 points)**

   Consider a portfolio that is long one European call option on a stock S with strike price K, and short one European call option on the same stock with strike price $L > K$. The expiration date of both options is $t = T$. Such a portfolio is called a *bull spread*. At time $t = T$ the payoff is

   $$(S_T - K)^+ - (S_T - L)^+.$$

   The one-period interest rate between $t = 0$ and $t = T$ is $r > 0$.

(a) Why must the value of the bull spread at time $t = 0$ be positive?

(b) Explain how to replicate this portfolio using a European put with strike price K and expiration date T, a European put with strike price L and expiration date T, and a loan or investment at the interest rate r. You may take a long or short position in either of the put options. [Hint: You may want to use the identity $(X - A)^+ - (A - X)^+ = X - A$.]

(c) Determine the value of the bull spread at $t = 0$ in terms of the values of the put options, P_0^K and P_0^L, the strike prices, K and L, and the interest rate r.

4. **(25 points)**

Consider a financial model with two times $t = 0$ and $t = 1$ and a one period interest rate $r > 0$. We say that a pair of strategies, $(X, \widetilde{X})$ is an Arb pair provided that

- Both strategies are self financing
- The initial capitals satisfy: $X_0 \leq \widetilde{X_0}$
- The terminal capitals satisfy $X_1 \geq \widetilde{X_1}$ for sure.
- With positive probability, the terminal capitals satisfy: $X_1 > \widetilde{X_1}$

Show that a model is arbitrage free if and only if it is free of Arb pairs. [Recall that an arbitrage is a strategy V that satisfies $V_0 = 0$, $V_1 \geq 0$ for sure, and $V_1 > 0$ with positive probability.]

5. **(25 points)** Omit.

6. **(25 points)** Consider a financial model with two trading times, $\{0, 1\}$, a single stock, S that pays no dividents, and a bank.

At $t = 0$ we can buy or sell any number of shares of the stock at the price $S_0 = \$40$ per share. At $t = 1$ the value of one share of stock will be either \$80 or \$20. The (one-period) interest rate for loans or deposits between $t = 0$ and $t = 1$ is $r = 25\%$.

X is a portfolio holding one call option on the stock with strike price $K^C = \$45$ and one put option on the stock with strike price $K^P = \$25$.

(a) Explain, without constructing a replicating portfolio, why $4 < X_0 < 28$. Be sure to fully justify any claims you make in your explanation.

(b) Find a replicating strategy and use it to determine X_0.

7. **(25 points)** Consider a financial model with three trading times, $\{0, 1, 2\}$ and two currencies, \$ and £. You are able to borrow or invest dollars at the one-period interest rate of $r^{\$} = 2\%$. You can borrow or invest pounds at the one period interest rate of $r^{£} = 3\%$. These rates are valid for investments between $t = 0$ and $t = 1$ or between $t = 1$ and $t = 2$. At $t = 0$ the spot exchange rate is $£1 = \$1.80$.

Consider also a contract that requires you to pay K dollars at $t = 1$ and receive $£1000$ at $t = 2$.

(a) Describe a replicating portflio for this contract. What is the value (in dollars) of this contract at $t = 0$?

(b) For what value of K will this contract have value zero at $t = 0$?

8. **(25 points)** Consider a financial model with four trading times, $\{0, 1, 2, 3\}$, a stock, S, and a bank. The stock is available at $t = 0$ at the price $S_0 = \$100$. Funds can be borrowed from or invested with the bank at the constant one-period interest rate $r = 10\%$.

   A derivative security, V, is available that, at time $t = 3$ pays the holder $S_2 - S_1$, the difference between the value of the stock at time $t = 2$ and $t = 1$.

   Describe a replicating portfolio and determine the arbitrage-free price of the security at $t = 0$, V_0.

9. **(25 points)** An investor creates a portfolio at $t = 0$ using the following assets:

   - Shares of a stock, S, that is available for sale or purchase at $t = 0$ for the price $S_0 = \$50$.
   - A put option on the stock with strike price $K_p = \$60$ and expiration date $t = T$. This put option is available at $t = 0$ for $P_0 = \$14$.
   - A call option on the stock with strike price $k_c = \$40$ and expiration date $t = T$. This call option is available at $t = 0$ for $C_0 = \$12$.
   - A forward contract on the stock, with execution date $t = T$ and forward price $\mathcal{F} = \$55$. This contract has value equal to zero at $t = 0$.

   The investor's portfolio holds 10 shares of the stock, 15 puts, a long position on 5 of the forward contract, and a short position on 10 of the calls.

   (a) What is the initial capital, X_0, of the investors portfolio?

   (b) If the price of the stock rises to $S_T = \$65$ per share, what will be the value, X_T, of the investors portfolio?

   (c) If the price of the stock falls to $S_T = \$35$ per share, what will be the value, X_T, of the investors portfolio?

10. **(25 points)** Omit.

11. **(25 points)** Consider a financial model with two times $\{0, 1\}$ and a stock, S, that pays no dividends. At $t = 0$ we can buy or sell any number of shares at the initial price $S_0 = \$44$. There is also a bank at which we can borrow or invest any amount at the one period interest rate $r = .25$. Additionally, there is a put option P on the stock with expiration date $T = 1$ and strike price $K_p = \$40$ which can be bought or sold in any quantity for $P_0 = \$4$.

    There are three possible outcomes ω_1, ω_2, ω_3 for the stock price:

$$S_1(\omega_1) = \$80; \quad S_1(\omega_2) = \$60; \quad S_1(\omega_3) = \$20;$$

Consider a call option on the stock with expiration date $T = 1$ and strike price $K_c = \$70$. Construct a replicating portfolio for the call option, and determine it's arbitrage-free price C_0 at $t = 0$.

12. **(25 points)** At $t = 0$ the following exchange rates are valid:

$$1 \text{ pound} = 2 \text{ dollars}$$

$$1 \text{ euro} = 1.5 \text{ dollars}$$

Between $t = 0$ and $t = 1$, dollars, pounds, and euros can be borrowed or invested at the interest rates

$$r^{\$} = 5\%, \quad r^{£} = 4\%, \quad r^{euros} = 7\%$$

An American investor arranges with a bank in London to receive, at time $t = 1$, capital in several different currencies in exchange for a payment (in pounds) at time $t = 0$.

If the investor is to receive \$10 and £10 and 10 Euros, what is the arbitrage-free value for the payment (in pounds) at $t = 0$?

13. **(25 points)** Consider a financial model that includes two trading times, $\{0, T\}$, an ideal bank with constant effective rate R, and a security V which can be bought or sold in any quantity at $t = 0$ for the price V_0.

Suppose that, under every possible outcome, the value V_T of the security at time $t = T$ satisfies

$$A < V_T < B.$$

Show that the arbitrage-free price V_0 of the security at $t = 0$ must satisfy

$$\frac{A}{(1+R)^T} < V_0 < \frac{B}{(1+R)^T}$$

14. **(25 points)** The price per share of a certain stock, S, is very high: $S_0 = \$10,000$. In order to facilitate trading in this stock, a bank has offered a contract that allows investors to divide the cost into multiple payments. The terms of the contract are:

- The investor makes payments of A dollars at times $t = 1$ and $t = 2$.
- At $t = 2$, the investors receives one share of the stock.
- No money is exchanged when the contract is entered at $t = 0$, and the amount A of the payments is agreed to at $t = 0$.

Assuming there is a bank at which any amount may be borrowed or invested at the constant effective rate of 5%, and that the stock makes no dividend payments, what is the arbitrage-free amount A for the payments?

Exam #1 Formula Sheet

Forward Exchange rates:

$$\mathcal{F}_{\mathcal{A}}^{\mathcal{B}} = \frac{1+r_A}{1+r_B} E_A^B.$$

Put payoff at maturity T:

$$P_T = (K - S_T)^+.$$

Call payoff at maturity T:

$$C_T = (S_T - K)^+.$$

No-arbitrage price of a fixed-income security

$$\mathcal{P} = \sum_{i=1}^{N} \frac{F_i}{(1+R_*(T_i))^{T_i}} = \sum_{i=1}^{N} F_i D(T_i) = \sum_{i=1}^{N} \frac{F_i}{(1+R_I)^{T_i}}$$