21-270 **Introduction to Mathematical Finance**

Week #8 Homework: Due Wednesday, March 4 **Spring 2020**

1. **Exercise 2.39**] Let $T_i = \frac{i}{2}$ for $i = 0, 1, 2, \ldots, 20$ and let $p_i[2]$ denote the nominal spot rate that will prevail at time T_i for borrowing and investing between times T_i and T_{i+1}. Consider a nonstandard interest rate swap between two parties A and B in which the notional principal will double half way through the swap. More precisely:

   - At each of the times T_i with $i = 1, 2, \ldots, 10$
   $$A \text{ pays } B \text{ the variable amount } F\frac{p_{i-1}[2]}{2} \text{ and}$$
   $$B \text{ pays } A \text{ the fixed amount } F\frac{q[2]}{2}.$$
   - At each of the times T_i with $i = 11, 12, \ldots, 20$
   $$A \text{ pays } B \text{ the variable amount } Fp_{i-1}[2] \text{ and}$$
   $$B \text{ pays } A \text{ the fixed amount } Fq[2].$$

   The (constant) swap rate $q[2]$ is chosen so that neither party pays anything to enter the agreement and the notional principal F is never paid. Find an expression for the swap rate $q[2]$ in terms of the discount factors $d(.5), (d(1), \ldots, d(10)$.

2. A coupon bond is being issued today. The face value of this bond is \$10,000, it makes 2 coupon payments per year, and it has maturity $T = 5$ years. The forward price to purchase this bond at $t = 2$ years, just after the coupon payment at that time, is $\mathcal{F}_{0,2} = \$9,708.95$.

   A zero coupon bond with face value \$1,000 and maturity $T = 5$ years can be bought or sold today for \$862.61.

   A zero coupon bond with face value \$1,000 and maturity $T = 2$ years can be bought or sold today for \$953.67.

   Today, the nominal annual yield to maturity of a 2-year annuity making 2 payments per year is $r_I^{(2)}[2] = 2.299\%$. The yield to maturity of a 5-year annuity making 2 payments per year is $r_I^{(5)}[2] = 2.691\%$.

   What is the nominal annual coupon rate $q[2]$ of the coupon bond?

3. [**Exercise 3.21**] Assume that there is a ideal money market with constant effective rate $R = .06$. Consider a stock S with current price per share $S_0 = \$48$. Find the arbitrage-free forward price for delivery of one share of stock in 10 months, assuming that

   (a) the stock will not pay any dividends during the next 10 months;

   (b) the stock will pay dividends twice during the next 10 months: a payment of \$.55 per share in 2 months and a payment of \$.60 cents per share in 8 months;

   (c) the stock will pay dividends twice during the next 10 months: each payment will be exactly .012 times the share price just prior to the dividend payment.

4. Some time ago, Aiofe took the long position on a forward contract for delivery of 100 ounces of gold one year from today at the forward price of \$1170. Forward contracts being written today for delivery one year from today have a forward price of \$1350. If the effective spot rate for deposits or loans starting today and ending one year from today is $R_*(1) = 2\%$, determine the value of Aiofe's position in the old contract.

5. [**Exercise 3.9**] A textile company in North Carolina will need 500,000 pounds of cotton two months from today. The company has two choices to obtain the cotton:

   (i) Purchase the cotton today at the spot price $S_0 = 45$ cents per pound and store it for two months. In order to store the cotton, the company will have to pay $\frac{1}{6}$ cents per pound at the end of each month that the cotton is in storage.

   (ii) Take a long position on a forward contract for delivery of the cotton in two months at the forward price of 46 cents per pound.

   Assuming that there is an ideal money market with effective annual rate $R = (1.01)^{12} - 1$, which choice should the company make? Discuss the assumptions needed to reach your conclusion.

6. [**Exercise 3.17**] A commodity swap (for N units) is an agreement made (at time 0) between two parties A and B under which A pays a fixed amount of cash to B at prescribed future dates and B pays A a variable amount of cash equal to N times the spot price of the designated commodity on each of those dates. Neither party pays anything to enter into the agreement. Let us assume that the swap dates are $T_i = \frac{i}{m}$, $i = 1, 2, \ldots, mn$ where m and n are given positive integers. At each of the times T_i, A pays the amount NF to B and B pays the amount NS_{T_i} to A, where S_t denotes the price of the commodity at time t. The swap price F, as well as the number of units N must be specified at time 0.

   (a) Find a formula for F in terms of the effective spot rates $R_*(T_i)$ and the forward prices $\mathcal{F}_{0,T_i}$ for delivery of one unit of the commodity at T_i.

   (b) A manufacturer of electrical equipment wants to enter a swap agreement (as party A) for copper with N =20,000 pounds and $m = n = 2$. Assuming that there is an ideal money market with constant effective rate $R = 5\%$ and that the forward prices (per pound) of copper are as follows:

   $$\mathcal{F}_{0,\frac{1}{2}} = \$2.795, \quad \mathcal{F}_{0,1} = \$2.74, \quad \mathcal{F}_{0,\frac{3}{2}} = \$2.685, \quad \mathcal{F}_{0,2} = \$2.64,$$

   determine the swap price F for one pound of copper.

   (c) Assume that the spot price of copper (at $t = 0$) is \$2.854 per pound. Even if the company in part (b) can store 80,000 pounds of copper at no cost, it will save money by entering the swap agreement as opposed to purchasing 80,000 pounds of copper at $t = 0$. Calculate the net present value (at $t = 0$) of the savings, assuming that the copper can be stored for free.