21-270 **Introduction to Mathematical Finance**

Homework #7 **Spring 2018**

1. [**Problem 2.16**] A former CMU student is working for a large investment bank. She has invented a new derivative security on shares of Amazon stock. The security has maturity one year and its payoff equals the average of the monthly prices of Amazon shares. In other words, at time $T = 1$ year the security will pay its holder the amount

$$\frac{1}{12}\sum_{i=1}^{12} S_{\frac{i}{12}},$$

where $S_{\frac{i}{12}}$ is the price of one share of Amazon at the end of the ith month. Compute the arbitrage-free price of the derivative security at $t = 0$. Assume that the initial price of one share of stock is $S_0 = \$100$, that Amazon stock does not pay dividends, and that there is an ideal money market with constant effective rate $R = (1.01)^{12} - 1$ (so that the nominal rate for monthly compounding is $r[12] = .12$.

2. Suppose there is an ideal bank with effective spot rate function satisfying $R_*(1) = .04$ and $R_*(2) = .07$. Consider the following two fixed income securities:

V^1: This security makes a payment of \$429.16 at $t = 1$ and a payment of \$100 at $t = 2$.

V^2: This security makes a payment of \$156.65 at $t = 1$ and a payment of \$400 at $t = 2$.

   (a) Determine the arbitrage-free price for each of these securities. Are they the same?

   (b) Determine the effective yield to maturity (internal rate of return) for each of these securities. Are they the same? Is this surprising?

3. You wish to purchase a home. You have found the ideal property, and negotiated a selling price of \$275,000 with the seller. You have saved \$55,000 that you will use for a down payment. You intend to finance the remaining \$220,000 with a 30-year fixed rate mortgage.

   (a) Your bank agrees to lend you \$220,000 with a mortgage rate of $r[12] = 5\%$. What will your monthly payment be?

   (b) But wait a minute, your bank also tells you there will have to pay an additional \$7,000 fee in order to take out the loan. "Don't worry," they say, "you can finance the closing costs." What will your monthly payment be if you borrow \$227,000 to cover the purchase and the closing costs?

   (c) Calculate the nominal rate $r_{APR}[12]$ that would result in the same monthly payment as in (b) for a loan of \$220,000 (i.e. without the closing costs). This rate is the *annual percentage rate* or *APR* for the loan. Banks are required to disclose this rate to help consumers compare loans with different rates and fee structures.

4. Consider an interest rate swap with maturity $T = 2$ with two swap dates per year. Given that $R_*(.5) = 2.5\%$, $R_*(1) = 3.5\%$, $R_*(1.5) = 4\%$, and $R_*(2) = 4.25\%$, find the swap rate $q^{swap}[2]$.

5. The 5-year effective spot rate is $R_*(5) = 3.85\%$ and annuities that have maturity 5 years and make monthly payments of \$100 are trading at \$5550. Find the swap rate $q^{swap}[12]$ for an interest rate swap having maturity 5 years and 12 swap dates per year.

6. (Exercise 2.29) Consider a coupon bond with face value IF and maturity $T = N$, for some positive integer N. Assume that the bond pays coupons m times per year at the nominal coupon rate $q[m]$ and that the current market price of the bond is $\hat{\mathcal{P}}$. Prove each of the following three statements:

   (a) The bond is trading at par (i.e. $\hat{\mathcal{P}} = F$) if and only if the nominal coupon rate is the same as the nominal yield to maturity (i.e., $q[m] = r_I[m]$).

   (b) The bond is trading above par (i.e. $\hat{\mathcal{P}} > F$) if and only if the nominal coupon rate is greater than the nominal yield to maturity (i.e., $q[m] > r_I[m]$).

   (c) The bond is trading below par (i.e. $\hat{\mathcal{P}} > F$) if and only if the nominal coupon rate is less than the nominal yield to maturity (i.e., $q[m] < r_I[m]$).

7. Let $T_i = \frac{i}{2}$ for $i = 0, 1, 2, \ldots, 20$ and let $p_i[2]$ denote the nominal spot rate that will prevail at time T_i for borrowing and investing between times T_i and T_{i+1}. Consider a nonstandard interest rate swap between two parties A and B in which the roles of receiveing fixed and floating payments will interchange half way through the swap. More precisely:

   - At each of the times T_i with $i = 1, 2, \ldots, 10$

   $$A \text{ pays } B \text{ the variable amount } F\frac{p_{i-1}[2]}{2} \text{ and}$$
   $$B \text{ pays } A \text{ the fixed amount } F\frac{q[2]}{2}.$$

   - At each of the times T_i with $i = 11, 12, \ldots, 20$

   $$A \text{ pays } B \text{ the fixed amount } F\frac{q[2]}{2} \text{ and}$$
   $$B \text{ pays } A \text{ the variable amount } F\frac{p_{i-1}[2]}{2}.$$

   The (constant) swap rate $q[2]$ is chosen so that neither party pays anything to enter the agreement (at time $t = 0$) and the notional principal F is never paid. Find an expression for the swap rate $q[2]$ in terms of the discount factors $d(.5), d(1), \ldots, d(10)$.