21-270 **Introduction to Mathematical Finance**

Homework #4: Due Wednesday, February 5. **Spring 2020**

1. Consider a financial model that includes a stock, S and a bank, and has two trading times $\{0, 1\}$.

   The bank will accept deposits or make loans at a one-period interest rate of $r = \frac{1}{4}$. An amount B_0 (positive for deposits, negative for loans) at time $t = 0$ will grow to $B_1 = (1 + r)B_0 = \frac{5}{4}B_0$ at time $t = 1$.

   At time $t = 0$, any number of shares of stock can be bought or sold at the price $S_0 = \$64$. At time one, the stock price will be one of two values, $S_1(H) = \$125$ or $S_1(T) = \$50$

   (a) Compute the initial prices P_0^{50}, P_0^{100}, and P_0^{150} of put options on the stock with strike prices 50, 100, and 150 respectively.

   (b) Compute the initial prices C_0^{50}, C_0^{100} and C_0^{150} of call options on the stock with strike prices 50, 100, and 150 respectively.

   (c) Compute the initial price $C_0^{50} - 2C_0^{100} + C_0^{150}$ of the butterfly spread created by trading in calls. Also compute the initial price $P_0^{50} - 2P_0^{100} + P_0^{150}$ of the butterfly spread created by trading in puts. Is it the same or different from the initial price of the butterfly spread created using calls? Why?

2. [**Exercise 1.14**] (Another Alternative Definition of Arbitrage). Consider a financial model with two times $t = 0$ and $t = 1$. Assume that there is a bank at which one can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the one period interest rate $r \geq 0$, where r is a constant that is known at time 0. Let us agree to say that a strategy is of type (Ar) provided that it is self-financing and the initial capital X_0 and terminal capital X_1 satisfy

   (i) $X_1 \geq (1 + r)X_0$ for sure;

   (ii) There is a strictly positive probability that $X_1 > (1 + r)X_0$

   Show that the model is arbitrage-free if and only if there are no strategies of type (Ar).

3. Consider a financial model with two times, $t = 0$ and $t = 1$, and two stocks S^1 and S^2 that pay no dividends. We can buy or sell any number of shares of each of the stocks at $t = 0$ at the initial prices $S_0^1 = S_0^2 = \$92$. There is also a bank at which we can borrow or invest any amount of money between $t = 0$ and $t = 1$ at the (one-period) interest rate $r = .25$. There are three possible outcomes ω_1, ω_2 and ω_3 regarding the stock prices, each having probability $\frac{1}{3}$. The possible stock prices at $t = 1$ are given by

$$S_1^1(\omega_1) = \$210, \quad S_1^1(\omega_2) = \$90, \quad S_1^1(\omega_3) = \$60,$$

$$S_1^2(\omega_1) = \$210, \quad S_1^2(\omega_2) = \$180, \quad S_1^2(\omega_3) = \$30.$$

Consider a derivative security V with payoff at $t = 1$ given by

$$V_1(\omega_i) = \max\{S_1^1(\omega_i), S_1^2(\omega_i)\}, \quad i = 1, 2, 3,$$

i.e. if outcome ω_i occcurs, the holder of the security receives the larger of $S_1^1(\omega_i)$ and $S_1^2(\omega_i)$ at $t = 1$. (This is an example of a *basket option.*) Let V_0 be the arbitrage-free price of V at $t = 0$.

(a) Explain why we know that $\$92 < V_0 < \frac{\$210}{1.25}$ without finding a replicating strategy.

(b) Find a replicating strategy and use it to determine V_0.

4. Consider a one-period binomial model that includes a stock, S, a bank, and has two trading times $\{0, 1\}$. The bank will accept deposits or make loans at a one-period interest rate of $r = .20$. An amount B_0 (positive for deposits, negative for loans) at time $t = 0$ will grow to $B_1 = (1 + r)B_0 = \frac{6}{5}B_0$ at time $t = 1$. At time $t = 0$, any number of shares of stock can be bought or sold at the price $S_0 = \$35$. At time one, the stock price will be one of two values, $S_1(H) = \$60$ or $S_1(T) = \$30$ There is a 2/3 probability that the value of the stock at $t = 1$ will be \$60, and a 1/3 probability that it will be \$30. (You may take it for granted that this model is free of arbitrage.)

   A client comes to you with \$100,000 to invest. She thinks the stock will increase in value, but is unwilling to incur *any* loss of capital. You you advise her to use all of her capital to purchase a derivative security that makes payments

$$\begin{cases} V_1(H) = A, & S_1 = 60 \\ V_1(T) = \$100,000, & S_1 = 30. \end{cases}$$

   (a) Without calculating the arbitrage-free value of A, one can see that to avoid arbitrage, the value of A must be more than \$120,000. Explain why.

   (b) Find the arbitrage-free value of A.

5. Consider a one-period trinomial model that includes a stock S, a put option P on the stock, and a bank. The bank will accept deposits or make loans at a one-period interest rate of $r = .25$. At time $t = 0$, any number of shares of stock can be bought or sold at the price $S_0 = \$32$. At time one, the stock price will take one of three values, $S_1(\omega_1) = \$60$, $S_1(\omega_2) = \$40$ or $S_1(\omega_3) = \$20$ At time $t = 0$ any number of put options on the stock with strike price $K = \$30$ and maturity $T = 1$ can be bought or sold at the price $P_0 = \$1.60$. You may take it for granted that this model is free of arbitrage.)

   Consider a derivative security V with payoff at $t = 1$ given by

$$V_1(\omega_i) = |S_1(\omega_i) - 40|$$

   for $i = 1, 2, 3$. (This security is called a *straddle option* on S with a strike price of \$40 and maturity $T = 1$) Find the arbitrage-free price V_0 of V at $t = 0$.