21-270 **Introduction to Mathematical Finance**

Homework #15 **Spring 2020**

1. Consider a complete one-period model with $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and let V^1, V^2, V^3, and V^4 denote the Arrow-Debru securities with payment functions

$$V_1^i(\omega) = \begin{cases} 1 & \omega = \omega_i \\ 0 & \omega \neq \omega_i. \end{cases}$$

Assume that the initial prices of these securities are

$$V_0^1 = \$0.38, \quad V_0^2 = \$0.095, \quad V_0^3 = \$0.19, \quad V_0^4 = \$0.285.$$

   (a) Find the arbitrage-free price W_0 of the derivative security W with payment function W_1 given by

$$W_1(\omega_1) = 14, \quad W_1(\omega_2) = 3, \quad W_1(\omega_3) = 0, \quad W_1(\omega_4) = -6.$$

   (b) Determine the interest rate r.

2. Consider a one-period binomial model with $u = 1.2$, $d = .95$, $r = .05$ and $\mathbb{P}(H) = \frac{1}{2}$, $\mathbb{P}(T) = \frac{1}{2}$.

   (a) Suppose that $U(x) = \sqrt{x}$, $x > 0$ and that $X_0 = \$1,000$. Fine the amount of money invested in stock at $t = 0$ and the amoutnt of money in the bank at $t = 0$ in the portfoio having the largest expected utility. (Use $S_0 = 100$ in your computations.)

   (b) Choose a percentage α, with $.2 < \alpha < .8$, that you feel would be a reasonable percentage of an initial investment X_0 to be put into a stock fund with a one-year investment horizon. (In other words, you will invest αX_0 in the stock and $(1-\alpha)X_0$ in the bank at $t = 0$ with the goal of maximizing your expected utility at time 1.)

   FInd a value $\beta > 0$ such that if the utility function

$$U(x) = \frac{-1}{x^\beta}, \quad x > 0$$

   is employed, then the optimal portfolio will have αX_0 invested in the stock at $t = 0$ and $(1-\alpha)X_0$ in the bank at $t = 0$.

3. Consider a one-period binomial model with $u = 1.2$, $d = .9$, $r = .1$ and $S_0 = 100$. An investor has utility function $U(x) = \ln x$ and initial capital $X_0 = 100$. Assume that the optimal strategy is to buy 1 share of stock (and invest nothing in the bank) at time 0. Determine $\mathbb{P}(H)$ and $\mathbb{P}(T)$.

4. Consider a complete, one-period model with $r = .1$, $\Omega = \{\omega_1, \omega_2, \omega_3\}$, and

$$\mathbb{P}(\omega_1) = \frac{1}{3}, \quad \mathbb{P}(\omega_2) = \frac{1}{3}, \quad \mathbb{P}(\omega_3) = \frac{1}{3},$$

and

$$\tilde{\mathbb{P}}(\omega_1) = \frac{1}{4}, \quad \tilde{\mathbb{P}}(\omega_2) = \frac{3}{8}, \quad \tilde{\mathbb{P}}(\omega_3) = \frac{3}{8}.$$

Suppose that an investor has initial capital $X_0 = 100$ and that the investor's utility function is

$$U(x) = \ln x, \quad x > 0.$$

Find the terminal capital $\hat{X}_1$ of the portfolio $\hat{X}$ that maximizes $\mathbb{E}[U(X_1)]$ over all portfolios having initial capital \$100.